Perturbed Inertial Krasnoselskii-Mann Iterations

and its application to image inpainting

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Optimisation and Fixed Point

Let $f\colon \mathcal{H} \to \mathbb{R}$ be differentiable.

Consider the problem: Find $\hat{x}$ such that

$$f(\hat{x}) = \min f(x)$$

Equivalent problem: Find $\hat{x}$ such that

$$\nabla f(\hat{x}) = 0$$

Other equivalent problem: Find $\hat{x}$ such that

$$T\hat{x} = \hat{x}, \quad \text{where } T = I - \nabla f$$

Optimisation and Fixed Point

Let $f, g \colon \mathcal{H} \to \mathbb{R}$. Suppose that f is differentiable.

Consider the problem: Find $\hat{x}$ such that

$$(f+g)(\hat{x}) = \min f(x) + g(x)$$

Equivalent problem: Find $\hat{x}$ such that

$$0 \in \partial(f+g)(\hat{x}) = \nabla f(\hat{x}) + \partial g(\hat{x})$$
$$\iff \hat{x} = (I - \partial g)^{-1}(I + \nabla f)(\hat{x})$$

Other equivalent problem: Find $\hat{x}$ such that

$$T\hat{x} = \hat{x}, \quad \text{where } T = (I - \partial g)^{-1}(I + \nabla f)$$

Fixed Point Iterative Schemes

Problem

For $T\colon \mathcal{H} \to \mathcal{H}$, find $\hat{x} \in \mathcal{H}$ such that $T\hat{x} = \hat{x}$.

Picard Iterations:

$$x_{k+1} = Tx_k$$

Krasnoselskii-Mann Iterations:

$$x_{k+1} = (1 - \lambda_k)x_k + \lambda_k Tx_k$$

Advantages of KM iterations over Picard iterations:

- Allows for a broader class of operators.
- Includes more diverse algorithms.

Acceleration Methods

Krasnoselskii-Mann Iterations:

$$x_{k+1} = (1-\lambda_k)x_k + \lambda_k Tx_k$$

Heavy-Ball Acceleration:

$$\begin{cases} y_k & = & x_k + \alpha_k(x_k - x_{k-1}) \\ x_{k+1} & = & (1-\lambda_k)y_k + \lambda_k Tx_k \end{cases}$$

Nesterov Acceleration:

$$\begin{cases} z_k & = & x_k + \beta_k(x_k - x_{k-1}) \\ x_{k+1} & = & (1-\lambda_k)z_k + \lambda_k Tz_k \end{cases}$$

General Inertial Scheme:

$$\begin{cases} y_k & = & x_k + \alpha_k(x_k - x_{k-1}) \\ z_k & = & x_k + \beta_k(x_k - x_{k-1}) \\ x_{k+1} & = & (1-\lambda_k)y_k + \lambda_k Tz_k \end{cases}$$

Adding Perturbations

Krasnoselskii-Mann Iterations:

$$x_{k+1} = (1-\lambda_k)x_k + \lambda_k T x_k$$

Perturbed Krasnoselskii-Mann Iterations:

$$x_{k+1} = (1-\lambda_k)x_k + \lambda_k T x_k + \varepsilon_k$$

Let $T_k = T + E_k$ where $E_k \to 0$. KM iterations with the operators T_k replacing T may be written as

$$\begin{aligned} x_{k+1} &= (1-\lambda_k)x_k + \lambda_k T_k x_k \\ &= (1-\lambda_k)x_k + \lambda_k T x_k + \lambda_k E_k x_k \\ &= (1-\lambda_k)x_k + \lambda_k T x_k + \varepsilon_k, \end{aligned}$$

with $\varepsilon_k = \lambda_k E_k x_k$.

Convergence Theorems

Perturbed General Inertial KM Iterations:

$$\begin{cases} y_k & = & x_k + \alpha_k(x_k - x_{k-1}) + \varepsilon_k \\ z_k & = & x_k + \beta_k(x_k - x_{k-1}) + \rho_k \\ x_{k+1} & = & (1-\lambda_k)y_k + \lambda_k T z_k + \theta_k \end{cases}$$

Theorem (Weak Convergence)

Let $T\colon \mathcal{H} \to \mathcal{H}$ be nonexpansive such that $F := \mathrm{Fix}(T) \neq \emptyset$. *Under mild conditions*, (x_k), (y_k) and (z_k) converge weakly to a same point in F.

Theorem (Strong Convergence)

Let $T\colon \mathcal{H} \to \mathcal{H}$ be contractive such that $\mathrm{Fix}(T) = \{p^*\}$. *Under mild conditions*, (x_k) converges strongly to p^*.

Extensions

The previous theorems may be extended by

1. replacing "nonexpansive" and "contractive" by "quasi-nonexpansive" ($K = 1$) and "quasi-contractive" ($K < 1$):

$$\|Tx - p\| \leq K\|x - p\|, \quad \forall x \in \mathcal{H}, p \in \mathrm{Fix}(T)$$

2. taking a family of operators $T_k \colon \mathcal{H} \to \mathcal{H}$ such that $F := \bigcap_{k \geq 1} \mathrm{Fix}(T_k) \neq \emptyset$ (Weak case) or $\mathrm{Fix}(T_k) = \{p^*\}$ for all $k \geq 1$ (Strong case).

3. taking a family of operators $T_k \colon \mathcal{H} \to \mathcal{H}$ such that $F := \mathrm{Li}(\mathrm{Fix}(T_k)) \neq \emptyset$ (Weak case) or $\mathrm{Fix}(T_k) = \{p_k\}$ such that $p_k \to p^*$ (Strong case).

Extended Convergence Theorems

Perturbed General Inertial KM Iterations:

$$\begin{cases} y_k &= x_k + \alpha_k(x_k - x_{k-1}) + \varepsilon_k \\ z_k &= x_k + \beta_k(x_k - x_{k-1}) + \rho_k \\ x_{k+1} &= (1-\lambda_k)y_k + \lambda_k T_k z_k + \theta_k \end{cases}$$

Extended Theorem (Weak Convergence)

Let $T_k \colon \mathcal{H} \to \mathcal{H}$ be quasi-nonexpansive such that $F := \mathrm{Li}(\mathrm{Fix}(T_k)) \neq \emptyset$. *Under mild conditions*, (x_k), (y_k) and (z_k) converge weakly to a same point in F.

Extended Theorem (Strong Convergence)

Let $T_k \colon \mathcal{H} \to \mathcal{H}$ be quasi-contractive such that $\mathrm{Fix}(T_k) = \{p_k\}$ and $p_k \to p^*$. *Under mild conditions*, (x_k) converges strongly to p^*.

Application to Optimisation

Problem

Let $f, g\colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$, $h\colon \mathcal{H} \to \mathbb{R}$, and $L\colon \mathcal{H} \to \mathcal{H}$. Find

$$\min_{x \in \mathcal{H}} f(x) + g(x) + h(Lx).$$

This may be solved by the **three-operator splitting method** (Davis, Yin, 2017):

$$T_k := I - \operatorname{prox}_{\rho_k g} + \operatorname{prox}_{\rho_k f} \circ (2\operatorname{prox}_{\rho_k g} - I - \rho_k L^* \circ \nabla h \circ L \circ \operatorname{prox}_{\rho_k g}).$$

$$\begin{cases} y_k &= x_k + \alpha_k(x_k - x_{k-1}) + \varepsilon_k \\ z_k &= x_k + \beta_k(x_k - x_{k-1}) + \rho_k \\ z_k^g &= \operatorname{prox}_{\rho_k g}(z_k) \\ z_k^f &= \operatorname{prox}_{\rho_k f}(2z_k^g - z_k - \rho_k L^* \circ \nabla h \circ L(z_k^g)) \end{cases}$$

Image Inpainting?

Figure: Not obtained through described algorithm!

Mathematical Formulation

$$\min_{Z \in [0,1]^{M \times N \times 3}} \left\{ \frac{1}{2} \| \mathcal{A} Z - Z_{\text{corrupt}} \|^2 + \sigma \| Z_{(1)} \|_* + \sigma \| Z_{(2)} \|_* \right\}$$

Visual Results

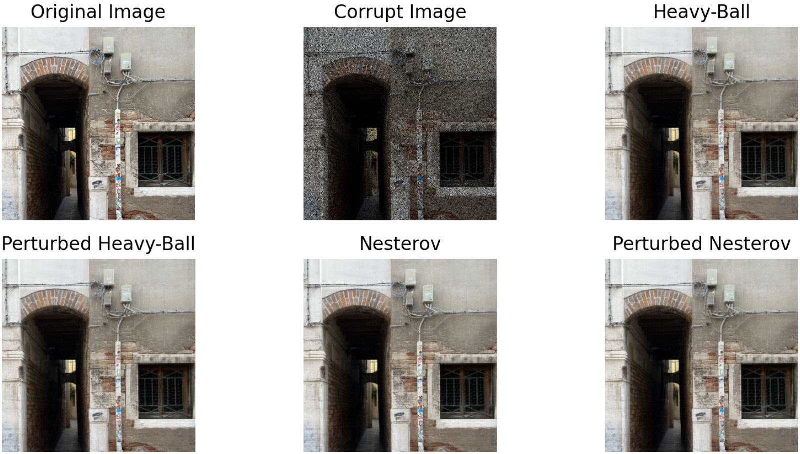

Figure: Process obtained with $\rho = 1.8$, $\lambda = 1.3$, and $\sigma = 0.5$.

Convergence Plots

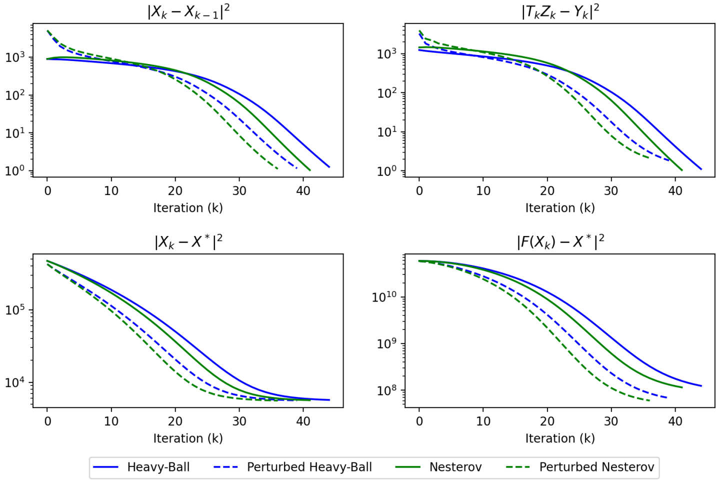

Figure: Process obtained with $\rho = 1.8$, $\lambda = 1.3$, and $\sigma = 0.5$.

Result Based on Algorithm

Figure: Obtained through perturbed inertial algorithm.

Conclusion

Summary:

- Perturbed Inertial KM Iterations generalise previously known algorithms.
- They incorporate multiple types of acceleration.
- They account for (rounding) errors and/or inexact computations.
- They also allow for operators not sharing a common fixed point.
- They converge weakly and strongly under mild conditions.
- They include the three-operator splitting method, used for optimisation problems such as the image inpainting problem.

Further possible research:

- Study of the rates of convergence.
- Why do the perturbations help?

Thank you!