Perturbed Inertial Krasnoselskii-Mann Iterations

and its applications to optimisation

Presenter: Daniel Cortild, *d.cortild@student.rug.nl*

Supervisors: Prof Dr J. G. Peypouquet & Prof Dr C. A. Bertoglio

13th of April 2023

university of groningen / faculty of science and engineering / mathematics and applied mathematics

Simple Fixed-Point Problem

Problem

For $T\colon \mathcal{H} \to \mathcal{H}$, find $\hat{x} \in \mathcal{H}$ such that $T\hat{x} = \hat{x}$.

Simple Fixed-Point Problem

Problem

For $T : \mathcal{H} \to \mathcal{H}$, find $\hat{x} \in \mathcal{H}$ such that $T\hat{x} = \hat{x}$.

Fixed-Point Iterations (Picard 1893, Lindelöf 1894):

$$x_{k+1} = Tx_k$$

Simple Fixed-Point Problem

Problem

For $T : \mathcal{H} \to \mathcal{H}$, find $\hat{x} \in \mathcal{H}$ such that $T\hat{x} = \hat{x}$.

Fixed-Point Iterations (Picard 1893, Lindelöf 1894):

$$x_{k+1} = Tx_k$$

Krasnoselskii-Mann Iterations (Mann 1953, Krasnoselskii 1955):

$$x_{k+1} = (1 - \lambda_k)x_k + \lambda_k Tx_k$$

Simple Fixed-Point Problem

Problem

For $T : \mathcal{H} \to \mathcal{H}$, find $\hat{x} \in \mathcal{H}$ such that $T\hat{x} = \hat{x}$.

Fixed-Point Iterations (Picard 1893, Lindelöf 1894):

$$x_{k+1} = Tx_k$$

Krasnoselskii-Mann Iterations (Mann 1953, Krasnoselskii 1955):

$$x_{k+1} = (1 - \lambda_k)x_k + \lambda_k Tx_k$$

Advantages of KM iterations over FP iterations:

Simple Fixed-Point Problem

Problem

For $T : \mathcal{H} \to \mathcal{H}$, find $\hat{x} \in \mathcal{H}$ such that $T\hat{x} = \hat{x}$.

Fixed-Point Iterations (Picard 1893, Lindelöf 1894):

$$x_{k+1} = Tx_k$$

Krasnoselskii-Mann Iterations (Mann 1953, Krasnoselskii 1955):

$$x_{k+1} = (1 - \lambda_k)x_k + \lambda_k Tx_k$$

Advantages of KM iterations over FP iterations:

- Includes more diverse algorithms.

Simple Fixed-Point Problem

Problem

For $T : \mathcal{H} \to \mathcal{H}$, find $\hat{x} \in \mathcal{H}$ such that $T\hat{x} = \hat{x}$.

Fixed-Point Iterations (Picard 1893, Lindelöf 1894):

$$x_{k+1} = Tx_k$$

Krasnoselskii-Mann Iterations (Mann 1953, Krasnoselskii 1955):

$$x_{k+1} = (1 - \lambda_k)x_k + \lambda_k Tx_k$$

Advantages of KM iterations over FP iterations:

- Includes more diverse algorithms.
- Allows for a broader class of operators.

Generalised Fixed-Point Problem

Problem

For $T_k \colon \mathcal{H} \to \mathcal{H}$, find $\hat{x} \in \mathcal{H}$ such that $T_k \hat{x} = \hat{x}$ for all $k \geq 1$.

Generalised Fixed-Point Problem

Problem

For $T_k \colon \mathcal{H} \to \mathcal{H}$, find $\hat{x} \in \mathcal{H}$ such that $T_k\hat{x} = \hat{x}$ for all $k \geq 1$.

Inertial KM Iterations (Fierro, Maulén, Peypouquet, 2022):

$$\begin{cases} y_k = x_k + \alpha_k(x_k - x_{k-1}) \\ x_{k+1} = (1-\lambda_k)y_k + \lambda_k T_k y_k \end{cases}$$

Generalised Fixed-Point Problem

Problem

For $T_k : \mathcal{H} \to \mathcal{H}$, find $\hat{x} \in \mathcal{H}$ such that $T_k \hat{x} = \hat{x}$ for all $k \geq 1$.

Inertial KM Iterations (Fierro, Maulén, Peypouquet, 2022):

$$\begin{cases} y_k & = & x_k + \alpha_k(x_k - x_{k-1}) \\ x_{k+1} & = & (1-\lambda_k)y_k + \lambda_k T_k y_k \end{cases}$$

Perturbed Inertial KM Iterations:

$$\begin{cases} y_k & = & x_k + \alpha_k(x_k - x_{k-1}) + \varepsilon_k \\ x_{k+1} & = & (1-\lambda_k)y_k + \lambda_k T_k y_k + \theta_k \end{cases}$$

Generalised Fixed-Point Problem

Problem

For $T_k : \mathcal{H} \to \mathcal{H}$, find $\hat{x} \in \mathcal{H}$ such that $T_k \hat{x} = \hat{x}$ for all $k \geq 1$.

Inertial KM Iterations (Fierro, Maulén, Peypouquet, 2022):

$$\begin{cases} y_k & = & x_k + \alpha_k(x_k - x_{k-1}) \\ x_{k+1} & = & (1-\lambda_k)y_k + \lambda_k T_k y_k \end{cases}$$

Perturbed Inertial KM Iterations:

$$\begin{cases} y_k & = & x_k + \alpha_k(x_k - x_{k-1}) + \varepsilon_k \\ x_{k+1} & = & (1-\lambda_k)y_k + \lambda_k T_k y_k + \theta_k \end{cases}$$

Advantages:

Generalised Fixed-Point Problem

Problem

For $T_k\colon \mathcal{H} \to \mathcal{H}$, find $\hat{x} \in \mathcal{H}$ such that $T_k\hat{x} = \hat{x}$ for all $k \geq 1$.

Inertial KM Iterations (Fierro, Maulén, Peypouquet, 2022):

$$\begin{cases} y_k &= \quad x_k + \alpha_k(x_k - x_{k-1}) \\ x_{k+1} &= \quad (1-\lambda_k)y_k + \lambda_k T_k y_k \end{cases}$$

Perturbed Inertial KM Iterations:

$$\begin{cases} y_k &= \quad x_k + \alpha_k(x_k - x_{k-1}) + \varepsilon_k \\ x_{k+1} &= \quad (1-\lambda_k)y_k + \lambda_k T_k y_k + \theta_k \end{cases}$$

Advantages:

- More stability

Generalised Fixed-Point Problem

Problem

For $T_k\colon \mathcal{H} \to \mathcal{H}$, find $\hat{x} \in \mathcal{H}$ such that $T_k\hat{x} = \hat{x}$ for all $k \geq 1$.

Inertial KM Iterations (Fierro, Maulén, Peypouquet, 2022):

$$\begin{cases} y_k & = & x_k + \alpha_k(x_k - x_{k-1}) \\ x_{k+1} & = & (1-\lambda_k)y_k + \lambda_k T_k y_k \end{cases}$$

Perturbed Inertial KM Iterations:

$$\begin{cases} y_k & = & x_k + \alpha_k(x_k - x_{k-1}) + \varepsilon_k \\ x_{k+1} & = & (1-\lambda_k)y_k + \lambda_k T_k y_k + \theta_k \end{cases}$$

Advantages:

- More stability
- Allows to relax certain conditions

Strong Convergence

Theorem

Let $T_k : \mathcal{H} \to \mathcal{H}$ be quasi-contractive such that $\mathrm{Fix}(T_k) = \{p^*\}$. Let (x_k) and (y_k) be generated by the perturbed inertial KM iterations. Then, *under mild conditions*, (x_k) converges strongly to p^*.

Strong Convergence

Theorem

Let $T_k \colon \mathcal{H} \to \mathcal{H}$ be quasi-contractive such that $\mathrm{Fix}(T_k) = \{p^*\}$. Let (x_k) and (y_k) be generated by the perturbed inertial KM iterations. Then, *under mild conditions*, (x_k) converges strongly to p^*.

Short sketch of proof:

Strong Convergence

Theorem

Let $T_k : \mathcal{H} \to \mathcal{H}$ be quasi-contractive such that $\mathrm{Fix}(T_k) = \{p^*\}$. Let (x_k) and (y_k) be generated by the perturbed inertial KM iterations. Then, *under mild conditions*, (x_k) converges strongly to p^*.

Short sketch of proof:

- Observe that for all $\varepsilon > 0$, eventually it holds that

$$\|x_k - p^*\|^2 \leq \alpha \|x_{k-1} - p^*\|^2 + \varepsilon.$$

Strong Convergence

Theorem

Let $T_k \colon \mathcal{H} \to \mathcal{H}$ be quasi-contractive such that $\mathrm{Fix}(T_k) = \{p^*\}$. Let (x_k) and (y_k) be generated by the perturbed inertial KM iterations. Then, *under mild conditions*, (x_k) converges strongly to p^*.

Short sketch of proof:

- Observe that for all $\varepsilon > 0$, eventually it holds that

$$\|x_k - p^*\|^2 \leq \alpha \|x_{k-1} - p^*\|^2 + \varepsilon.$$

- Iterate in order to conclude.

Strong Convergence

Theorem

Let $T_k \colon \mathcal{H} \to \mathcal{H}$ be quasi-contractive such that $\mathrm{Fix}(T_k) = \{p^*\}$. Let (x_k) and (y_k) be generated by the perturbed inertial KM iterations. Then, *under mild conditions*, (x_k) converges strongly to p^*.

Short sketch of proof:

- Observe that for all $\varepsilon > 0$, eventually it holds that

$$\|x_k - p^*\|^2 \leq \alpha \|x_{k-1} - p^*\|^2 + \varepsilon.$$

- Iterate in order to conclude.

Corollary

Let $T_k \colon \mathcal{H} \to \mathcal{H}$ be quasi-contractive such that $\mathrm{Fix}(T_k) = \{p_k\}$, with $p_k \to p^*$. Let (x_k) and (y_k) be generated by the perturbed inertial KM iterations. Then, *under mild conditions*, (x_k) converges strongly to p^*.

Weak Convergence

Theorem

Let $T_k \colon \mathcal{H} \to \mathcal{H}$ be quasi-nonexpansive such that $F := \bigcap_{k \geq 1} \mathrm{Fix}(T_k) \neq \emptyset$. Let (x_k) and (y_k) be generated by the perturbed inertial KM iterations. Then, *under mild conditions*, both (x_k) and (y_k) converge weakly to a same point in F.

Weak Convergence

Theorem

Let $T_k\colon \mathcal{H} \to \mathcal{H}$ be quasi-nonexpansive such that $F \coloneqq \bigcap_{k\geq 1} \mathrm{Fix}(T_k) \neq \emptyset$. Let (x_k) and (y_k) be generated by the perturbed inertial KM iterations. Then, *under mild conditions*, both (x_k) and (y_k) converge weakly to a same point in F.

Short sketch of proof:

Weak Convergence

Theorem

Let $T_k \colon \mathcal{H} \to \mathcal{H}$ be quasi-nonexpansive such that $F := \bigcap_{k \geq 1} \operatorname{Fix}(T_k) \neq \emptyset$. Let (x_k) and (y_k) be generated by the perturbed inertial KM iterations. Then, *under mild conditions*, both (x_k) and (y_k) converge weakly to a same point in F.

Short sketch of proof:

- The weak limit points of (x_k) and of (y_k) coincide.

Weak Convergence

Theorem

Let $T_k \colon \mathcal{H} \to \mathcal{H}$ be quasi-nonexpansive such that $F := \bigcap_{k \geq 1} \mathrm{Fix}(T_k) \neq \emptyset$. Let (x_k) and (y_k) be generated by the perturbed inertial KM iterations. Then, *under mild conditions*, both (x_k) and (y_k) converge weakly to a same point in F.

Short sketch of proof:

- The weak limit points of (x_k) and of (y_k) coincide.
- Every weak limit point of (y_k) (and thus of (x_k)) is in F.

Weak Convergence

Theorem

Let $T_k\colon \mathcal{H} \to \mathcal{H}$ be quasi-nonexpansive such that $F := \bigcap_{k\geq 1} \mathrm{Fix}(T_k) \neq \emptyset$. Let (x_k) and (y_k) be generated by the perturbed inertial KM iterations. Then, *under mild conditions*, both (x_k) and (y_k) converge weakly to a same point in F.

Short sketch of proof:

- The weak limit points of (x_k) and of (y_k) coincide.
- Every weak limit point of (y_k) (and thus of (x_k)) is in F.
- For all $p \in F$, $\|x_k - p\|$ is convergent.

Weak Convergence

Theorem

Let $T_k\colon \mathcal{H} \to \mathcal{H}$ be quasi-nonexpansive such that $F \coloneqq \bigcap_{k\geq 1} \mathrm{Fix}(T_k) \neq \emptyset$. Let (x_k) and (y_k) be generated by the perturbed inertial KM iterations. Then, *under mild conditions*, both (x_k) and (y_k) converge weakly to a same point in F.

Short sketch of proof:

- The weak limit points of (x_k) and of (y_k) coincide.
- Every weak limit point of (y_k) (and thus of (x_k)) is in F.
- For all $p \in F$, $\|x_k - p\|$ is convergent.
- Conclude by Opial's Lemma.

Weak Convergence

Theorem

Let $T_k \colon \mathcal{H} \to \mathcal{H}$ be quasi-nonexpansive such that $F := \bigcap_{k \geq 1} \mathrm{Fix}(T_k) \neq \emptyset$. Let (x_k) and (y_k) be generated by the perturbed inertial KM iterations. Then, *under mild conditions*, both (x_k) and (y_k) converge weakly to a same point in F.

Short sketch of proof:

- The weak limit points of (x_k) and of (y_k) coincide.
- Every weak limit point of (y_k) (and thus of (x_k)) is in F.
- For all $p \in F$, $\|x_k - p\|$ is convergent.
- Conclude by Opial's Lemma.

Corollary

Let $T_k \colon \mathcal{H} \to \mathcal{H}$ be quasi-nonexpansive such that $F := \mathrm{Li}\ \mathrm{Fix}(T_k) \neq \emptyset$. Let (x_k) and (y_k) be generated by the perturbed inertial KM iterations. Then, *under mild conditions*, both (x_k) and (y_k) converge weakly to a same point in F.

Application to Optimisation

Problem

Let $f, g\colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$, $h\colon \mathcal{H} \to \mathbb{R}$, and $L\colon \mathcal{H} \to \mathcal{H}$ satisfy certain conditions. Find

$$\min_{x \in \mathcal{H}} f(x) + g(x) + h(Lx).$$

Application to Optimisation

Problem

Let $f, g\colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$, $h\colon \mathcal{H} \to \mathbb{R}$, and $L\colon \mathcal{H} \to \mathcal{H}$ satisfy certain conditions. Find

$$\min_{x \in \mathcal{H}} f(x) + g(x) + h(Lx).$$

This may be solved by a perturbed inertial version of the **three-operator splitting method** (Davis, Yin, 2017):

$$\begin{cases} y_k &= x_k + \alpha_k(x_k - x_{k-1}) + \varepsilon_k \\ x_k^g &= \operatorname{prox}_{\rho g}(y_k) \\ x_k^f &= \operatorname{prox}_{\rho f}(2x_k^g - y_k - \rho L^* \circ \nabla h \circ L(x_k^g)) \\ x_{k+1} &= y_k + \lambda_k(x_k^f - x_k^g) + \theta_k \end{cases}$$

Image Inpainting?

Original Image

Image Inpainting?

Original Image

Corrupt Image

Image Inpainting?

Image Inpainting?

Figure: Not obtained through described algorithm!

Image Inpainting?

Figure: Not obtained through described algorithm!

Mathematical Formulation

$$\min_{Z \in [0,1]^{M \times N \times 3}} \left\{ \frac{1}{2} \| \mathcal{A}Z - Z_{\text{corrupt}} \|^2 + \sigma \| Z_{(1)} \|_* + \sigma \| Z_{(2)} \|_* \right\}$$

Visual Results

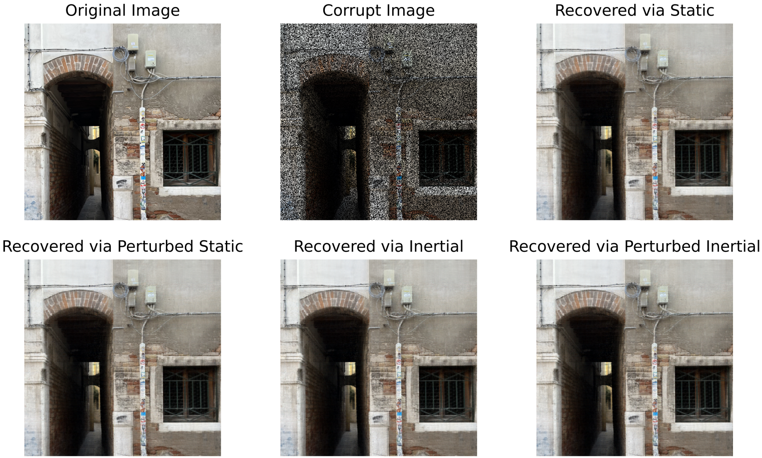

Figure: Process obtained with $\rho = 2$, $\lambda = 0.8$, and $\sigma = 0.25$.

Result Based on Algorithm

Figure: Obtained through perturbed inertial algorithm.

Conclusion

Today we learnt that:

Conclusion

Today we learnt that:

- Perturbed Inertial KM Iterations generalise previously known algorithms.

Conclusion

Today we learnt that:

- Perturbed Inertial KM Iterations generalise previously known algorithms.
- They account for (rounding) errors and/or inexact computations.

Conclusion

Today we learnt that:

- Perturbed Inertial KM Iterations generalise previously known algorithms.
- They account for (rounding) errors and/or inexact computations.
- They also allow for operators not sharing a common fixed point.

Conclusion

Today we learnt that:

- Perturbed Inertial KM Iterations generalise previously known algorithms.
- They account for (rounding) errors and/or inexact computations.
- They also allow for operators not sharing a common fixed point.
- They converge weakly and strongly under mild conditions.

Conclusion

Today we learnt that:

- Perturbed Inertial KM Iterations generalise previously known algorithms.
- They account for (rounding) errors and/or inexact computations.
- They also allow for operators not sharing a common fixed point.
- They converge weakly and strongly under mild conditions.
- They include the three-operator splitting method, used for optimisation problems.

Conclusion

Today we learnt that:

- Perturbed Inertial KM Iterations generalise previously known algorithms.
- They account for (rounding) errors and/or inexact computations.
- They also allow for operators not sharing a common fixed point.
- They converge weakly and strongly under mild conditions.
- They include the three-operator splitting method, used for optimisation problems.

Further possible research:

Conclusion

Today we learnt that:

- Perturbed Inertial KM Iterations generalise previously known algorithms.
- They account for (rounding) errors and/or inexact computations.
- They also allow for operators not sharing a common fixed point.
- They converge weakly and strongly under mild conditions.
- They include the three-operator splitting method, used for optimisation problems.

Further possible research:

- Study of the rate of convergence of various residual quantities in case of weak convergence.

Conclusion

Today we learnt that:

- Perturbed Inertial KM Iterations generalise previously known algorithms.
- They account for (rounding) errors and/or inexact computations.
- They also allow for operators not sharing a common fixed point.
- They converge weakly and strongly under mild conditions.
- They include the three-operator splitting method, used for optimisation problems.

Further possible research:

- Study of the rate of convergence of various residual quantities in case of weak convergence.
- Study of the rate of convergence in case of strong convergence

Main Litterature

BAUSCHKE, H. H., AND COMBETTES, P. L.
Convex Analysis and Monotone Operator Theory in Hilbert Spaces.
CMS Books in Mathematics. Springer New York, NY, February 2017.

DAVIS, D., AND YIN, W.
A Three-Operator Splitting Scheme and its Optimization Applications.
Set-Valued and Variational Analysis 25 (June 2017), 829–858.

FIERRO, I., MAULÉN, J. J., AND PEYPOUQUET, J.
Inertial Krasnoselskii-Mann Iterations, October 2022.

LI, J., LI, M., AND FAN, H.
Image Inpainting Algorithm Based on Low-Rank Approximation and Texture Direction.
Mathematical Problems in Engineering (December 2014).

PEYPOUQUET, J.
Convex Optimization in Normed Spaces: Theory, Methods and Examples.
SpringerBriefs in Optimization. Springer Cham, March 2015.