Krasnoselskii-Mann Iterations

Inertia, Perturbations and Approximation

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Fixed Point Iterative Schemes

Problem

Given a Hilbert space $\mathcal{H}$ and an operator $T\colon \mathcal{H} \to \mathcal{H}$, find $\hat{x} \in \mathcal{H}$ such that $T\hat{x} = \hat{x}$.

Picard Iterations:

$$x_{k+1} = Tx_k$$

Converges linearly in the contractive setting, might not converge in the nonexpansive setting.

Krasnoselskii-Mann Iterations:

$$x_{k+1} = (1 - \lambda_k)x_k + \lambda_k Tx_k$$

Converges in both the contractive and the nonexpansive setting.

Acceleration Methods

Heavy-Ball Acceleration:

$$\begin{cases} y_k &= x_k + \alpha_k(x_k - x_{k-1}) \\ x_{k+1} &= (1-\lambda_k)y_k + \lambda_k T x_k \end{cases}$$

Nesterov Acceleration:

$$\begin{cases} z_k &= x_k + \beta_k(x_k - x_{k-1}) \\ x_{k+1} &= (1-\lambda_k)z_k + \lambda_k T z_k \end{cases}$$

General Inertial Scheme:

$$\begin{cases} y_k &= x_k + \alpha_k(x_k - x_{k-1}) \\ z_k &= x_k + \beta_k(x_k - x_{k-1}) \\ x_{k+1} &= (1-\lambda_k)y_k + \lambda_k T z_k \end{cases}$$

Inexactness and Diagonalization

Perturbed Krasnoselskii-Mann Iterations:

$$x_{k+1} = (1-\lambda_k)x_k + \lambda_k T x_k + \varepsilon_k$$

Diagonal Krasnoselskii-Mann Iterations:

$$x_{k+1} = (1-\lambda_k)x_k + \lambda_k T_k x_k$$

General Scheme:

$$\begin{cases} y_k &= x_k + \alpha_k(x_k - x_{k-1}) + \varepsilon_k \\ z_k &= x_k + \beta_k(x_k - x_{k-1}) + \rho_k \\ x_{k+1} &= (1-\lambda_k)y_k + \lambda_k T_k z_k + \theta_k \end{cases}$$

Weak Convergence

Theorem

Let $T_k \colon \mathcal{H} \to \mathcal{H}$ be a family of nonexpansive operators and assume

- $F := \bigcap_{k \geq 1} \operatorname{Fix}(T_k) \neq \emptyset$,
- $(I - T_k)$ is asymptotically demiclosed,
- $(\varepsilon_k), (\rho_k), (\theta_k) \in \ell^1(\mathcal{H})$.

Then, under *mild assumptions* on the parameters, $x_k \rightharpoonup p^*$ with $p^* \in F$.

Notes;

- If $T_k \equiv T$, the first two conditions are redundant.
- We do not assume (x_k) to be bounded, the residuals to be summable, or any similar assumption.

Shifting Operators

Instead of $\bigcap_{k\geq 1} \mathrm{Fix}(T_k) \neq \emptyset$, assume $F_\infty \coloneqq \mathrm{Li}\,\mathrm{Fix}(T_k) \neq \emptyset$ and fix any $p_k \in \mathrm{Fix}(T_k)$ and $p_\infty \in F_\infty$ such that $p_k \to p_\infty$.

By setting $\tilde{T}_k x = T_k(x + p_k - p_\infty) - p_k + p_\infty$, $\tilde{x}_k = x_k - p_{k-1} + p_\infty$, $\tilde{y}_k = y_k - p_k + p_\infty$, and $\tilde{z}_k = z_k - p_k + p_\infty$, our algorithm becomes

$$\begin{cases} \tilde{y}_k & = & \tilde{x}_k + \alpha_k(\tilde{x}_k - \tilde{x}_{k-1}) + \tilde{\varepsilon}_k, \\ \tilde{z}_k & = & \tilde{x}_k + \beta_k(\tilde{x}_k - \tilde{x}_{k-1}) + \tilde{\rho}_k, \\ \tilde{x}_{k+1} & = & (1-\lambda_k)\tilde{y}_k + \lambda_k \tilde{T}_k \tilde{z}_k + \theta_k, \end{cases}$$

for

$$\begin{cases} \tilde{\varepsilon}_k \coloneqq p_{k-1} - p_k + \alpha_k(p_{k-1} - p_{k-2}) + \varepsilon_k, \\ \tilde{\rho}_k \coloneqq p_{k-1} - p_k + \beta_k(p_{k-1} - p_{k-2}) + \rho_k. \end{cases}$$

Weak Convergence

Let $T_k \colon \mathcal{H} \to \mathcal{H}$ be a family of nonexpansive operators.

Theorem

Assume

- $F := \bigcap_{k \geq 1} \operatorname{Fix}(T_k) \neq \emptyset$,
- $(I - T_k)$ is asymptotically demiclosed,
- $(\varepsilon_k), (\rho_k), (\theta_k) \in \ell^1(\mathcal{H})$.

Then, under *mild assumptions* on the parameters, $x_k \rightharpoonup p^*$ with $p^* \in F$.

Corollary

Assume

- $F_\infty := \operatorname{Li} \operatorname{Fix}(T_k) \neq \emptyset$,
- (T_k) nicely approximates F_∞,
- $(\varepsilon_k), (\rho_k), (\theta_k) \in \ell^1(\mathcal{H})$,
- there exists a sequence (p_k) with $p_k \in \operatorname{Fix}(T_k)$ and $(p_k - p_{k-1}) \in \ell^1(\mathcal{H})$.

Then, under *mild assumptions* on the parameters, $x_k \rightharpoonup p^*$ with $p^* \in F_\infty$.

Strong Convergence

Let $T_k : \mathcal{H} \to \mathcal{H}$ be a family of q_k-contractive operators with $q_k \leq q < 1$.

Theorem

Assume

- $\mathrm{Fix}(T_k) = \{p^*\}$,
- $(\varepsilon_k), (\rho_k), (\theta_k) \in \ell^2(\mathcal{H})$.

Then, under *mild assumptions* on the parameters, $x_k \to p^*$.

Moreover, if $\varepsilon_k \equiv \rho_k \equiv \theta_k \equiv 0$, the convergence becomes linear.

Corollary

Assume

- $\mathrm{Fix}(T_k) = \{p_k\}$ with $p_k \to p^*$ and $(p_k - p_{k-1}) \in \ell^2(\mathcal{H})$,
- $(\varepsilon_k), (\rho_k), (\theta_k) \in \ell^2(\mathcal{H})$,

Then, under *mild assumptions* on the parameters, $x_k \to p^*$.

Image Inpainting

Figure: Not obtained through described algorithm!

Mathematical Formulation

$$\min_{Z \in [0,1]^{M \times N \times 3}} \left\{ \frac{1}{2} \|\mathcal{A}Z - Z_{\mathrm{corrupt}}\|^2 + \sigma \|Z_{(1)}\|_* + \sigma \|Z_{(2)}\|_* \right\}$$

Visual Results

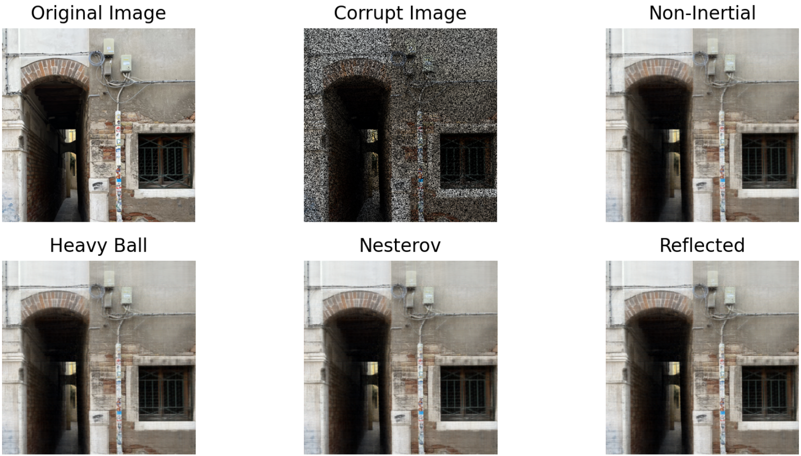

Figure: Process obtained with $\rho = 1.8$, $\lambda = 1.3$, and $\sigma = 0.5$.

Convergence Plots

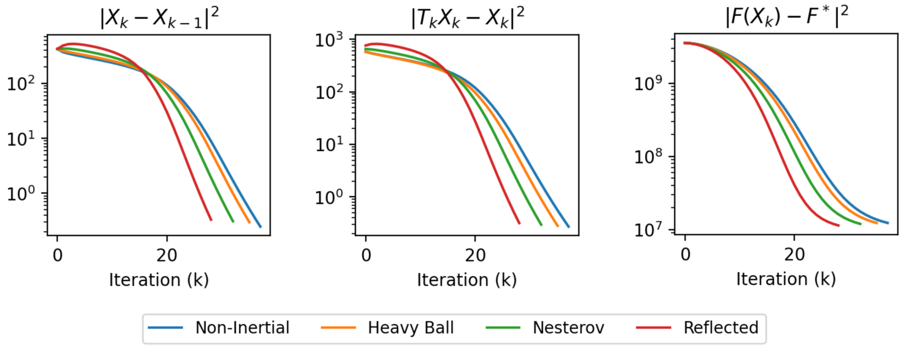

Figure: Process obtained with $\rho = 1.8$, $\lambda = 1.3$, and $\sigma = 0.5$.

Result Based on Algorithm

Figure: Obtained through Nesterov accelerated algorithm.

Further Research Directions:

- Optimal parameter selection
- Contribution of perturbation parameters

Paper: Cortild, D. & Peypouquet, J. (2024). Krasnoselskii-Mann Iterations: Inertia, Perturbations and Approximation. *arXiv preprint arXiv:2401.16870*

Thank you!