Krasnoselskii-Mann Iterations

Inertia, Perturbations and Approximation

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Fixed Point Iterative Schemes

Problem

Given a Hilbert space $\mathcal{H}$ and an operator $T\colon \mathcal{H} \to \mathcal{H}$, find $\hat{x} \in \mathcal{H}$ such that $T\hat{x} = \hat{x}$.

Picard Iterations:

$$x_{k+1} = Tx_k$$

Convergence linearly in the contractive setting, might not converge in the nonexpansive setting.

Krasnoselskii-Mann Iterations:

$$x_{k+1} = (1 - \lambda_k)x_k + \lambda_k Tx_k$$

Converges in both the contractive and the nonexpansive setting.

Acceleration Methods

Heavy-Ball Acceleration:

$$\begin{cases} y_k & = & x_k + \alpha_k(x_k - x_{k-1}) \\ x_{k+1} & = & (1-\lambda_k)y_k + \lambda_k T x_k \end{cases}$$

Nesterov Acceleration:

$$\begin{cases} z_k & = & x_k + \beta_k(x_k - x_{k-1}) \\ x_{k+1} & = & (1-\lambda_k)z_k + \lambda_k T z_k \end{cases}$$

General Inertial Scheme:

$$\begin{cases} y_k & = & x_k + \alpha_k(x_k - x_{k-1}) \\ z_k & = & x_k + \beta_k(x_k - x_{k-1}) \\ x_{k+1} & = & (1-\lambda_k)y_k + \lambda_k T z_k \end{cases}$$

Inexactness and Diagonalization

Perturbed Krasnoselskii-Mann Iterations:

$$x_{k+1} = (1 - \lambda_k)x_k + \lambda_k Tx_k + \varepsilon_k$$

Diagonal Krasnoselskii-Mann Iterations:

$$x_{k+1} = (1 - \lambda_k)x_k + \lambda_k T_k x_k$$

General Scheme:

$$\begin{cases} y_k & = & x_k + \alpha_k(x_k - x_{k-1}) + \varepsilon_k \\ z_k & = & x_k + \beta_k(x_k - x_{k-1}) + \rho_k \\ x_{k+1} & = & (1 - \lambda_k)y_k + \lambda_k T_k z_k + \theta_k \end{cases}$$

Weak Convergence

Theorem (Weak Convergence)

Let $T_k : \mathcal{H} \to \mathcal{H}$ be a family of nonexpansive operators such that $F := \bigcap_{k \geq 1} \mathrm{Fix}(T_k) \neq \emptyset$ and such that $(I - T_k)$ is asymptotically demiclosed. Assume *mild conditions* on the parameters, and assume that the error sequences $(\varepsilon_k), (\rho_k), (\theta_k) \in \ell^1(\mathcal{H})$. Then (x_k, y_k, z_k) converges weakly to (p^*, p^*, p^*) with $p^* \in F$.

Theorem (Weak Convergence, Extended Version)

Let $T_k : \mathcal{H} \to \mathcal{H}$ be a family of nonexpansive operators such that $F_\infty := \mathrm{Li}\,\mathrm{Fix}(T_k) \neq \emptyset$ and such that (T_k) nicely approximates F_∞. Assume there exists a sequence (p_k) with $p_k \in \mathrm{Fix}(T_k)$ and $(p_k - p_{k-1}) \in \ell^1(\mathcal{H})$. Assume *mild conditions* on the parameters, and assume that the error sequences $(\varepsilon_k), (\rho_k), (\theta_k) \in \ell^1(\mathcal{H})$. Then (x_k, y_k, z_k) converges weakly to (p^*, p^*, p^*) with $p^* \in F_\infty$.

Strong Convergence

Theorem (Strong Convergence)

Let $T_k \colon \mathcal{H} \to \mathcal{H}$ be a family of q_k-contractive operators with $q_k \leq q < 1$ and $\mathrm{Fix}(T_k) = \{p^*\}$. Assume *mild conditions* on the parameters, and assume that the error sequences $(\varepsilon_k), (\rho_k), (\theta_k) \in \ell^2(\mathcal{H})$. Then (x_k) converges strongly to p^*.

Moreover, if $\varepsilon_k \equiv \rho_k \equiv \theta_k \equiv 0$, the convergence becomes linear.

Theorem (Strong Convergence, Extended Version)

Let $T_k \colon \mathcal{H} \to \mathcal{H}$ be a family of q_k-contractive operators with $q_k \leq q < 1$, and $\mathrm{Fix}(T_k) = \{p_k\}$ with $p_k \to p^*$ and $(p_k - p_{k-1}) \in \ell^2(\mathcal{H})$. Assume *mild conditions* on the parameters, and assume that the error sequences $(\varepsilon_k), (\rho_k), (\theta_k) \in \ell^2(\mathcal{H})$. Then (x_k, y_k, z_k) converges strongly to (p^*, p^*, p^*).

Image Inpainting

Figure: Not obtained through described algorithm!

Mathematical Formulation

$$\min_{Z \in [0,1]^{M \times N \times 3}} \left\{ \frac{1}{2} \|\mathcal{A}Z - Z_{\text{corrupt}}\|^2 + \sigma \|Z_{(1)}\|_* + \sigma \|Z_{(2)}\|_* \right\}$$

Visual Results

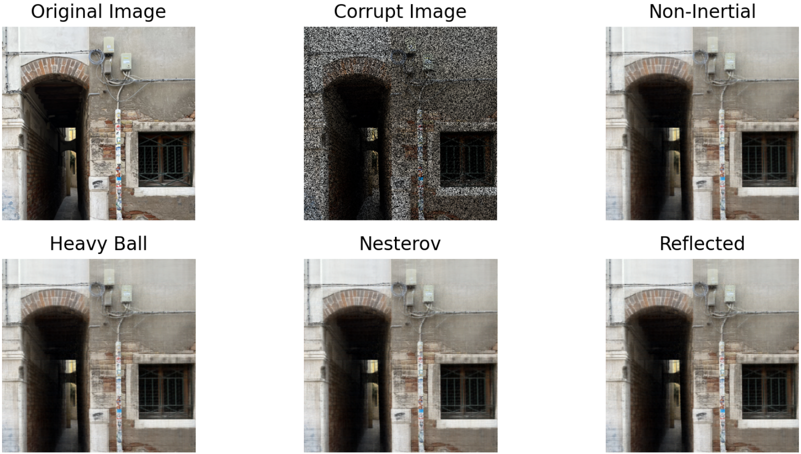

Figure: Process obtained with $\rho = 1.8$, $\lambda = 1.3$, and $\sigma = 0.5$.

Convergence Plots

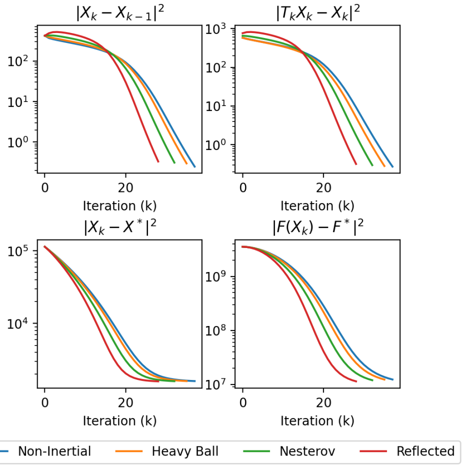

Figure: Process obtained with $\rho = 1.8$, $\lambda = 1.3$, and $\sigma = 0.5$.

Result Based on Algorithm

Figure: Obtained through Nesterov accelerated algorithm.

Thank you!

Paper: Cortild, D. & Peypouquet, J. (2024). Krasnoselskii-Mann Iterations: Inertia, Perturbations and Approximation. *arXiv preprint arXiv:2401.16870*