Regularization Methods for Hierarchical Variational Inequalities

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Regularization in Hierarchical Optimization

Consider the **Hierarchical Optimization (HO)** Problem

$$\min \{g(x) \text{ s.t. } x \in \operatorname{argmin} f\} \quad \longrightarrow \quad \min \left\{ f(x) + \beta g(x) + \frac{\alpha}{2}\|x - w\|^2 \right\}$$

Tikhonov Regularization

Why? As $\beta \to 0$, forces solution to be in $\operatorname{argmin} f$.

How to pick β?

1. Iteratively solve the problem for smaller β.
2. Update β during algorithm (diagonal scheme).

Proximal Term

Why? Enforces strong convexity for $\alpha > 0$. **But** shifts solution.

How to pick it?

1. Make $\alpha \to 0$, but then we lose strong convexity.
2. Make the anchor w track the solution.

Hierarchical Variational Inequalities

Consider a **Variational Inequality (VI)**

$$\text{Find } u \in \mathbf{SOL}(\mathbf{VI}(G, K)) \quad = \quad \{u_* : \langle Gu_*, v - u_* \rangle \geq 0 \quad \forall v \in K\} \quad = \quad \mathbf{Zer}(G + N_K)$$

In this talk, we focus on **Hierarchical VIs (HVI)**

$$\text{Find } u \in \mathbf{SOL}(\mathbf{VI}(G, S_0)) \quad \text{where } S_0 = \mathbf{SOL}(\mathbf{VI}(F, K)) = \mathbf{Zer}(F + N_K).$$

We consider the regularized problem as

$$\text{Find } u = u(\alpha, \beta, w) \in \mathbf{Zer}(F + N_K + \beta G + \alpha(I - w)), \tag{P-Reg}$$

with $\beta > 0$ a Tikhonov term and $\alpha > 0$ a proximal term.

Fixed-Point (FP) Encoding Assumption. For all (α, β, w), we assume having access to a **contractive** FP encoding $T = T(\alpha, \beta, w)$ of Problem (P-Reg), namely that $\{u\} = Z(\mathbf{Fix}(T))$ for some **nonexpansive** Z.

Proposed Algorithm

Algorithm 1 Diagonal Equilibrium Tracking Method

Given:

- $w_0 \in \mathcal{H}$ given initial point;
- $\alpha > 0$ a proximal term;
- $(\beta_n) \subset \mathbb{R}_{>0}$ a sequence of Tikhonov parameters;
- $N \geq 1$ a given number of restarts.

for $n = 0, \dots, N-1$ **do**
  Set $T_n = T(\alpha, \beta_n, w_n)$ and $\{p_n\} = \mathrm{Fix}(T_n)$.
  Determine a point v_n such that $\|v_n - p_n\| \leq \varepsilon_n$.
  Let $w_{n+1} = v_n$ denote the next anchor point.
end for
return $\overline{w}_N$ (weighted average)

Determining the point $v_n \in \mathcal{H}$ may be done, for instance, through a Picard iteration. This takes $\mathcal{O}(\log(1/\varepsilon_n))$ sub-iterations.

Gap Function Definitions

Recall the problem:

$$\text{Find } u \in \mathrm{SOL}(\mathrm{VI}(G, S_0)) \quad \text{where } S_0 = \mathrm{SOL}(\mathrm{VI}(F, K)) = \mathrm{Zer}(F + N_K).$$

Define

$$\mathrm{Gap}_{\mathrm{feas}}(x) = \sup_{(v,y)\in \mathrm{gr}(F+N_K)} \langle v, x - y\rangle,$$

and

$$\mathrm{Gap}_{\mathrm{opt}}(u) = \sup_{\mathrm{Zer}(F+N_K)} \langle Gv, u - v\rangle.$$

It holds that:

- $\mathrm{Gap}_{\mathrm{feas}}(x) \geq 0$ for all $x \in \mathcal{H}$ and $\mathrm{Gap}_{\mathrm{feas}}(x) \leq 0 \iff x \in S_0$;
- If $u \in S_0$, $\mathrm{Gap}_{\mathrm{opt}}(u) \leq 0 \iff u \in \mathrm{SOL}(\mathrm{VI}(G, S_0))$.

Goal: Determine $x \in \mathcal{H}$ such that $\mathrm{Gap}_{\mathrm{feas}}(x) = 0$ and $\mathrm{Gap}_{\mathrm{opt}}(x) \leq 0$.

Convergence Results

Theorem (G Monotone Case)

Let $\beta_n = (n+1)^{-b}$ for $b \in (0, 0.5)$ and $\varepsilon_n = o(\beta_n)$. For $N \geq 1$,

$$0 \leq \mathrm{Gap}_{\mathrm{feas}}(\overline{w}_N) \leq \mathcal{O}\left(1/N^{b}\right), \quad \mathrm{Gap}_{\mathrm{opt}}(\overline{w}_N) \leq \mathcal{O}\left(1/N^{1-b}\right).$$

In particular, all weak accumulation points of $(\overline{w}_N)$ are solutions.

Theorem (G μ-Strongly Monotone Case)

Let $\beta_n = \frac{\alpha}{2\mu n+\xi}$ for $\xi > 0$ and $\varepsilon_n = o(\beta_n)$. For $N \geq 1$,

$$0 \leq \mathrm{Gap}_{\mathrm{feas}}(\overline{w}_N) \leq \mathcal{O}\left(\log(N)/N\right), \quad \mathrm{Gap}_{\mathrm{opt}}(\overline{w}_N) \leq \mathcal{O}\left(1/N\right).$$

In particular, all weak accumulation points of $(\overline{w}_N)$ are solutions.

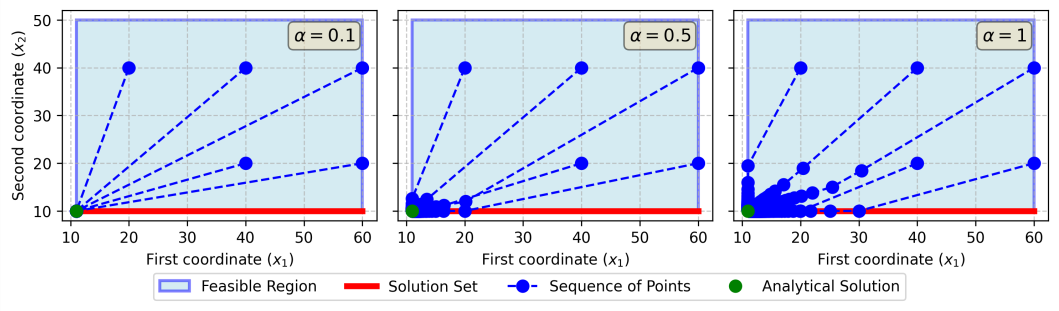
Recall: All complexity results suffer from a logarithmic term due to the inner loop.

Numerical Example: Equilibrium Selection (1/3)

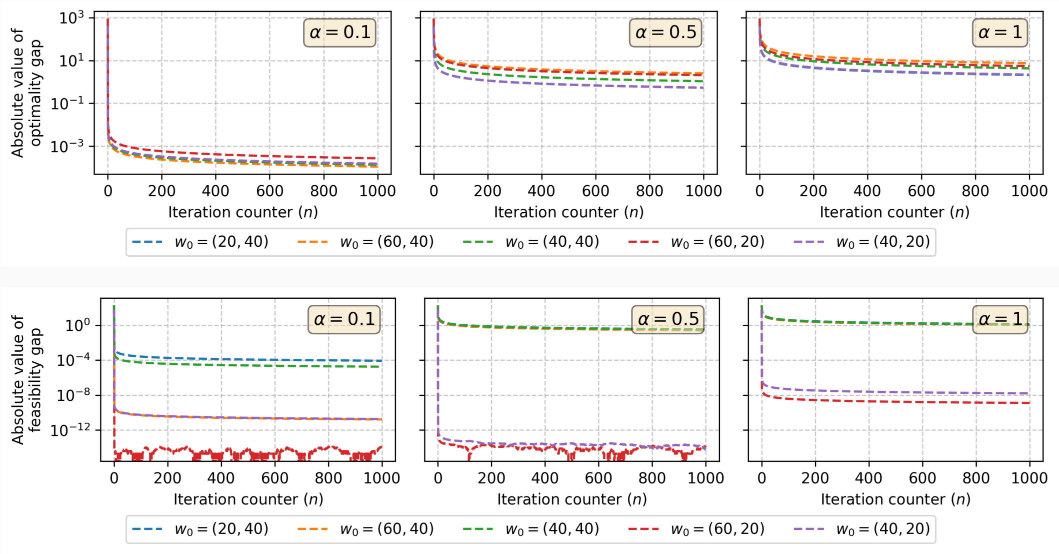
Consider the two-player zero-sum game:

$$\begin{cases} \min_{x_1} 20 - 0.1x_1x_2 + x_1 \\ \text{s.t. } x_1 \in X_1 = [11, 60] \end{cases} \qquad \begin{cases} \min_{x_2} -20 + 0.1x_1x_2 - x_1 \\ \text{s.t. } x_2 \in X_2 = [10, 50]. \end{cases}$$

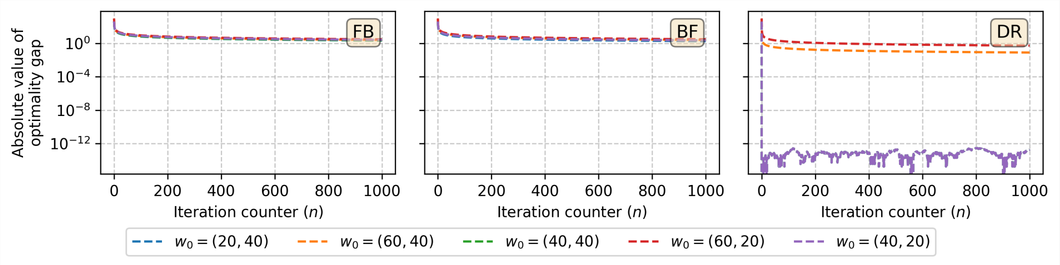
We aim to find the minimal-norm solution. This matches Problem (HVI) with $G = I$, $F = (\nabla_{x_1} f, -\nabla_{x_2} f)$ and $K = X_1 \times X_2$.

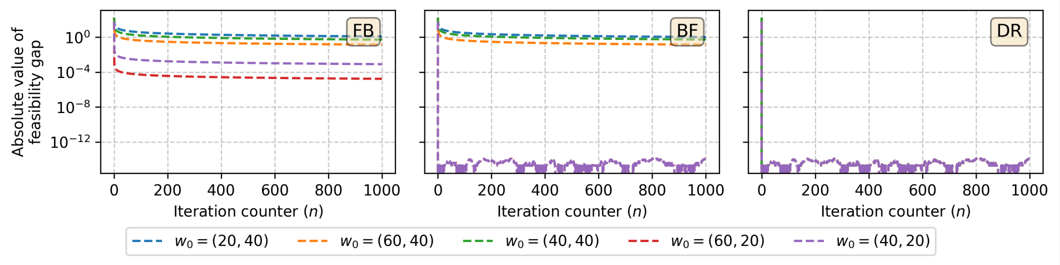


Numerical Example: Equilibrium Selection (2/3)



Numerical Example: Equilibrium Selection (3/3)





Conclusion

Summary

- We consider a diagonal regularization scheme with tracking to solve Hierarchical VIs.
- We obtain convergence and rate guarantees.
- Our method is flexible, in the sense that it can adapt to the structure of the problem.

Outlook

- Remove the logarithmic factor induced by the inner loop.
- Extend the analysis to nonexpansive FP encodings.

Based on: Daniel Cortild, Meggie Marschner, and Mathias Staudigl. *Regularization Methods for Solving Hierarchical Variational Inequalities with Complexity Guarantees.* arXiv preprint arXiv:2512.20772. Dec. 2025. arXiv: `2512.20772 [math]`