Probabilistic Methods

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1. Introduction

The key idea behind probabilistic methods is that, in order to show there exists an object with a certain property, we show that a random object satisfies this property with positive probability. This idea will be formalized and extended to more complicated situations, and illustrated on numerous examples. We will introduce purely probability-based arguments, arguments based on expected values, arguments based on alterations, and arguments based on inequalities such as Markov's Inequality or the Local Lemma. No prerequisites are required, as the necessary notions and tools will be introduced at the start of each session. After some short theory, the classes will mainly be focused on problem solving.

1.1. Structure

The notes are structured as follows:

2. Probabilities & Expectation

2.1. Some Theory

Disclaimer: The following section is very informal is by not meant to be taken as rigorous definitions. If you wish to learn more about the formal definitions of probabilities, please look up a book on measure theory.

- We define a **random variable** to be a quantity that takes random values. This could for instance be the outcome of a standard six-sided dice, where we call the random variable D.
- A **random event** is something that is true when the event occurs, and false when it does not. This could for instance be the event that the dice shows the value 6, or mathematically speaking, that $D=6$. Let us denote this random event by $E=\{D=6\}$.
- We can define the **probability** of a random event as the odds that said random event will be true. This will be denoted by $\mathbb{P}(\cdot)$. For instance, $\mathbb{P}(E)=\mathbb{P}(D=6)=1/6$. Note how we omitted the $\{\}$ when writing out the definition of the event E within $\mathbb{P}$, this is simply for notational convenience. Note that it always holds that $0\leq\mathbb{P}(E)\leq 1$.
- The probability of an event occuring may be computed as the ratio of the number of times it occurs, over the total number of possibilities. If a random variable A takes values in Ω, then any event may be written as $E=\{A\in\mathcal{A}\}$ for some $\mathcal{A}\subset\Omega$. Then it holds that $\mathbb{P}(E)=|\mathcal{A}|/|\Omega|$.

- We say two events A and B are **independent** when the fact whether A occurs or not does not influence the probability of B occurring, and vice-versa. For instance, the events of rolling a 6 on a dice, following by landing a heads on a coin toss are independent. However, the events of rolling a 6 and rolling an even number are not independent, since knowing one will influence the probability of the other occurring (if a 6 is rolled the outcome is automatically even, and if we know an even number if being rolled, the probability of it being a 6 is now 1/3 instead of 1/6).
- We can combine events to form new events. The standard "operations" on events would be **and** ($\cap$) and **or** ($\cup$), which are interpreted exactly as one would think. For instance, if A is the event that our dice rolled a multiple of 2, and B is the event that our dice rolled a multiple of 3, then $A \cap B$ is the event that the dice rolled a 6, and $A \cup B$ is the event that the dice did not roll a 1 or a 5.
- Another way of viewing independent events is through their **joint probability**. In fact, two events A and B are independent if, and only if, $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$. This is interpreted as; The probability that both events happen is the same as the probability of the first happening, followed by the second happening. Of course this generalizes to an arbitrary number of independent events A_i, as $\mathbb{P}(\cap_{i=1}^{n} A_i) = \prod_{i=1}^{n} \mathbb{P}(A_i)$.
- A similar rule to the previous one holds for general events (independent or not); For any two events A and B, the **sum rule** holds. That is $\mathbb{P}(A \cup B) + \mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B)$. If we again take our example of rolling a multiple of 2 (event A) and rolling a multiple of 3 (event B), then it is easily verified that $\mathbb{P}(A \cup B) + \mathbb{P}(A \cap B) = \frac{4}{6} + \frac{1}{6} = \frac{3}{6} + \frac{2}{6} = \mathbb{P}(A) + \mathbb{P}(B)$.
- A bi-product of the sum rule is that for any two events A and B, $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$. This can be generalized into **Boole's Inequality**, stating that $\mathbb{P}(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i)$, where A_i are random events, not necessarily independent.
- We say two events A and B are **disjoint** if A and B cannot happen at the same time, that is $A \cap B = \emptyset$. In specific, $\mathbb{P}(A \cap B) = 0$ (but note this is not necessarily equivalent).
- If the events $A_1, \ldots, A_n$ are pairwise disjoint, Boole's Inequlality becomes an equality, namely $\mathbb{P}(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i)$.
- Although this is a subcase of Boole's Inequality, we also introduce the **Union Bound**, which states that for any events $A_1, \ldots, A_n$, if $\mathbb{P}(A_1) + \ldots + \mathbb{P}(A_n) < 1$, then there is a nonzero probability that none of the events occur.

We complement the above definitions by introducing a new concept, namely the one of **expectation** or **expected value**.

- Given a random variable E that takes values in a set Ω, we define its **expectation** (or **expected value**) as
$$\mathbb{E}[E] = \sum_{\omega \in \Omega} \mathbb{P}(E = \omega) \cdot \omega.$$
Note that the sum of replaced by an integral in the case where Ω is a continuous set. We shall not study this setting here though. For instance, back to our example with a dice roll, if E records the value rolled, then
$$\mathbb{E}[E] = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = \frac{7}{2}.$$
The expectation of a random variable is interpreted as its **average value**.

- The most important property about expectations is that they are **linear**, in the sense that for any two events A and B, $\mathbb{E}[A+B]=\mathbb{E}[A]+\mathbb{E}[B]$. This can easily be shown using the definition of the expectation and some algebra, yielding

$$\begin{aligned}\mathbb{E}[A+B] &= \sum_{x\in\Omega}\sum_{y\in\Omega}(x+y)\mathbb{P}(A=x,B=y)\\ &= \sum_{x\in\Omega}x\sum_{y\in\Omega}\mathbb{P}(A=x,B=y)+\sum_{y\in\Omega}y\sum_{x\in\Omega}\mathbb{P}(A=x,B=y)\\ &= \sum_{x\in\Omega}x\mathbb{P}(A=x)+\sum_{y\in\Omega}y\mathbb{P}(B=y)=\mathbb{E}[A]+\mathbb{E}[B].\end{aligned}$$

  Of course the generalizes to an arbitrary number of events.

- An important realization is that if $\mathbb{E}[E]=\overline{E}$, then E takes values e_1 and e_2 satisfying $e_1\le\overline{E}\le e_2$ with strictly positive probability. This is interpreted as: An event can be larger and smaller than its average value.

- Recall an **event** of a property is a random variable that takes value 1 if the given property holds, and 0 elsewise. By the definition of expectation, it holds that

$$\mathbb{E}[A]=\mathbb{P}(A=1)=\mathbb{P}(\text{Property } P \text{ holds}),$$

  where A is the event that property P holds.

2.2. Idea behind method

The idea behind **zeroth order probabilistic methods** is the following:

- The problem we are confronted with with looks as follows: Prove that amongst all possible $x\in\mathcal{X}$, where $\mathcal{X}$ is *some set* of possible objects, there exists an object $x\in\mathcal{X}$ such that $P(x)$ holds true, where P is some property.

- Instead of constructing such an x, we select a $X\in\mathcal{X}$ at random, and prove that $\mathbb{P}(P(X))>0$. This shows there must be some x such that $P(x)$ holds true.

- Alternatively, we can also show that $\mathbb{P}(\neg P(X))<1$, where $\neg P(X)$ denotes that P is false. This results in the same conclusion.

- Sometimes it will be easier to split $P(X)$ into smaller parts. If $P(X)$ holds true if, and only if, a lot of small simple properties $P_i(X)$ holds true, then we know $P(X)=\cup_i P_i(X)$. Then by Boole's Inequality we know that

$$\mathbb{P}(P(X))\le\sum_i\mathbb{P}(P_i(X)),$$

  which is often simpler to compute. If the events $P_i(X)$ are pairwise disjoint, the inequality is an equality!

The idea behind **first order probabilistic methods** is the following:

- The problem we are confronted with looks as follows: Prove that amongst all possible $x\in\mathcal{X}$, where $\mathcal{X}$ is *some set* of possible objects, there exists an object $x\in\mathcal{X}$ that satisfies $P(x)\ge c$, for some function (or property) P and some constant c.

- Instead of building such an $x \in \mathcal{X}$, or proving by contradiction that such an $x \in \mathcal{X}$ must exist, we shall use a **first order probabilistic method**. Indeed, we shall select an $X \in \mathcal{X}$ **randomly**, such that $P(X)$ is also random.
- If we then prove that $\mathbb{E}[P(X)] \geq c$, then we know that there exists an object $x \in \mathcal{X}$ such that $P(x) \geq c$. This follows the idea that there always exists an object that is higher than the average.
- Of course the same reasoning applies to proving there exists an object $x \in \mathcal{X}$ such that $P(x) \leq c$.
- If c is an integer and we know that $P(x)$ is always an integer (for instance, if $x \in \mathcal{X}$ is a graph and $P(x)$ counts something in this graph), then proving that $\mathbb{E}[P(X)] > c-1$ is sufficient to conclude that there exists an object $x \in \mathcal{X}$ satisfying $P(x) \geq c$.

2.3. Exercises on Expectation

Exercise 2.1 (HMMT 2006)**.** At a nursery, 2006 babies sit in a circle. Suddenly, each baby randomly pokes either the baby to its left or to its right. What is the expected value of the number of unpoked babies?

Exercise 2.2 (AHSME 1989)**.** Suppose that 7 boys and 13 girls line up in a row. Let S be the number of places in the row where a boy and a girl are standing next to each other. For example, for the row GBBGGGBGBGGGBGBGGBGG we have $S = 12$. Find the expected value of S.

Exercise 2.3 (NIMO 4.3)**.** One day, a bishop and a knight were on squares in the same row of an infinite chessboard, when a huge meteor storm occurred, placing a meteor in each square on the chessboard independently and randomly with probability p. Neither the bishop nor the knight were hit, but their movement may have been obstructed by the meteors. For what value of p is the expected number of valid squares that the bishop can move to (in one move) equal to the expected number of squares that the knight can move to (in one move)?

Exercise 2.4 (IMO 2001/4)**.** Let n be an odd integer strictly greater than 1 and let $c_1, \ldots, c_n$ be positive integers. For each permutation $\pi = (\pi_1, \ldots, \pi_n)$ of $\{1, \ldots, n\}$, define $S(\pi) = \sum_{i=1}^{n} c_i \pi_i$. Prove there exist two permutations π and μ such that $S(\pi) - S(\mu)$ is divisible by $n!$.

2.4. Classical Examples

Example 2.1 (Ramsey Numbers)**.** Let n and k be positive integers satisfying $k \geq 3$ and $n \leq 2^{k/2}$. We denote by K_n the complete graph on n vertices, that is the graph $K_n = ([n], \binom{[n]}{2})$, where $\binom{[n]}{2}$ is the set of all subsets of size 2 of $[n]$. Prove that it is possible to colour the edges of K_n in blue and red such that no subset of k vertices form a monochromatic clique (all edges between these k vertices have the same colour).

Proof. Given a set $S \subset [n]$ with k vertices, it contains $\binom{k}{2} = \frac{k(k-1)}{2}$ edges in between them. Since each edge is randomly assigned, the probability that each edge is the same colour is

$$\mathbb{P}(E_S = 1) = \underbrace{(1/2)^{\binom{k}{2}}}_{\mathbb{P}\text{ is red}} + \underbrace{(1/2)^{\binom{k}{2}}}_{\mathbb{P}\text{ is blue}} = (1/2)^{\binom{k}{2}-1},$$

and since E_S takes values either in 0 or in 1, it holds that

$$\mathbb{E}[E_S] = \mathbb{P}(E_S = 1) = (1/2)^{\binom{k}{2}-1}.$$

Note that there are a total of $\binom{n}{k}$ subsets of size k in $[n]$, and hence

$$\mathbb{E}[E] = \mathbb{E}\left[\sum_{|S|=k} E_S\right] = \sum_{|S|=k} \mathbb{E}[E_S] = \binom{n}{k} \cdot (1/2)^{\binom{k}{2}-1}.$$

We can now conclude using the approximation $\binom{n}{k} \leq \frac{n^k}{k!}$ and the assumption that $n \leq 2^{k/2}$:

$$\mathbb{E}[E] = \binom{n}{k} \cdot (1/2)^{\binom{k}{2}-1} < \frac{n^k}{k!}(1/2)^{\frac{k(k-1)}{2}-1} \leq \frac{2^{k^2/2}}{k!}(1/2)^{\frac{k(k-1)}{2}-1} = \frac{2^{k/2+1}}{k!},$$

which one easily checks is less than 1 for $k \geq 3$. □

Example 2.2 (Erdos 1963)**.** A *hypergraph* is a pair $H = (V, E)$ where V is a finite set, called *vertices*, and E is a set of subsets of V, called *edges*. We say a hypergraph is *n-uniform* if each edge contains precisely n vertices. Note that a 2-uniform hypergraph is a just a standard graph.

We say a hypergraph is *k-colourable* if we can colour each vertex using one of k colours, such that no edge has all its vertices of the same colour (such an edge is called monochromatic).

Prove that every n-uniform hypergraph with strictly less than 2^{n-1} edges is 2-colourable.

Proof. Colour the vertices randomly in red and blue with probability 0.5. For an edge $e \in E$, define the event A_e that the edge is monochromatic. Then $\mathbb{P}(A_e) = 2^{1-n}$, and hence, by Boole's Inequality,

$$\mathbb{P}\left(\bigcup_{e \in E} A_e\right) \leq \sum_{e \in E} \mathbb{P}(A_e) = |E| \cdot 2^{1-n} < 1.$$

So with strictly positive probability none of A_e occur, so there exists a colouring in which no edge is monochromatic, so H is 2-colourable. □

2.5. Problems

Problem 2.1 (Romania 2004)**.** Prove that for any complex numbers $z_1, \ldots, z_n \in \mathbb{C}$ satisfying $|z_1|^2 + \cdots + |z_n|^2 = 1$, there exist $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$ such that

$$\left|\sum_{k=1}^{n} \varepsilon_k z_k\right| \leq 1.$$

Problem 2.2. A tournament T is an orientation on the complete graph $K_n = \left([n], \binom{[n]}{2}\right)$, where an oriented edge (a, b) represents that a beats b. We say that a tournament T has property S_k if, for any set of k people, there is a person that beats all people in the set.

If $\binom{n}{k}(1 - 2^{-k})^{n-k} < 1$, prove there exists a tournament on n vertices that has property S_k.

Problem 2.3 (MOP Test 2007/7/1)**.** Let $n \in \mathbb{N}$. Arrange in a $n^2 \times n^2$ matrix the numbers $1, 2, \ldots, n^2$ each exactly n^2 times. Show there exists a row or a column with a least n distinct numbers.

Problem 2.4 (EGMO 2019/5)**.** Let $n \geq 2$ be an integer, and let $a_1, a_2, \ldots, a_n$ be positive integers. Show that there exist positive integers $b_1, \ldots, b_n$ that satisfy:

1. $a_i \leq b_i$ for all $i = 1, \ldots, n$,
2. The values of $b_1, \ldots, b_n$ are all distinct modulo n,
3. $b_1 + \cdots + b_n \leq n\left(\frac{n-1}{2} + \left\lfloor \frac{a_1 + \cdots + a_n}{n} \right\rfloor\right)$.

Problem 2.5 (Kraft's inequality)**.** A word of length n, denoted $w = w_1 \cdots w_n$, is a sequence of characters (w_i) from a finite alphabet. We say a word $w = w_1 \cdots w_n$ is a **prefix** of a word $v = v_1 \cdots v_m$ if $n \leq m$ and $w_i = v_i$ for all $1 \leq i \leq n$.

We let $\mathcal{F}$ be a (possibly infinite) collection of words over an alphabet of size r, and we let N_i denote the number of words in $\mathcal{F}$ of length i. We suppose that $\sum_{i=1}^{\infty} N_i \cdot r^{-i} > 1$. Prove that there exist two distinct words $w, v \in \mathcal{F}$ such that w is a prefix of v.

Problem 2.6 (SJSU M179 Midterm)**.** Let $K_{n,n}$ be a graph with vertex sets $V_1 \cup V_2$ where $|V_1| = |V_2| = n$ and edge set $E = \{(a,b) \colon a \in V_1, b \in V_2\}$. Let G be any subgraph of $K_{n,n}$, with at least $n^2 - n + 1$ edges. Prove that G has a perfect matching, that is one can form n pairs of vertices with one in V_1 and one in V_2, such that the edge between each pair is present in G.

Problem 2.7 (IMO Shortlist 2006/C3)**.** Let S be a finite set of points in the plane such that no three of them are colinear. Denote by $\mathcal{P}$ the set of convex polygons whose vertices are in S, and $P \in \mathcal{P}$, define $a(P)$ to be its number of vertices and $b(P)$ to be the number of points of S not contained in P. Show that, for all $x \in \mathbb{R}$,

$$\sum_{P \in \mathcal{P}} x^{a(P)} (1-x)^{b(P)} = 1.$$

Problem 2.8 (Bollobas 1965)**. (*)** Let X be any set and let $\mathcal{F} = \{(A_i, B_i)\}_{i=1}^{h}$ be a set of pairs of subsets of X (so $A_i, B_i \subset X$ for all $i = 1, \ldots, h$). We say that $\mathcal{F}$ is a (k,l)**-system** if $|A_i| = k$, $|B_i| = l$ and $A_i \cap B_i = \emptyset$ for all $i = 1, \ldots, h$, and that $A_i \cap B_j \neq \emptyset$ for all distinct $1 \leq i, j \leq h$.

Prove that if $\mathcal{F}$ is a (k,l)-system, then $h \leq \binom{k+l}{k}$.

Problem 2.9 (MOP Test 2008/7/2)**. (**)** Suppose that $a, b, c \in \mathbb{R}_{>0}$ such that, for all $n \in \mathbb{N}$, $\lfloor an \rfloor + \lfloor bn \rfloor = \lfloor cn \rfloor$. Show that at least one of a, b, c is an integer.

Problem 2.10 (Erdos 1965)**. (**)** Prove that every set of n nonzero integers contains a sum-free subset of size larger than $\frac{n}{3}$.

3. Alterations

3.1. Some Theory

- We cannot always expect that a random object will be "good enough" on average, which is the key behind the probabilistic method. This occurs for instance when the number of "desirable outcomes" is very small compared to all the possible outcomes. One then needs to apply some tricks to make the probabilistic method work.

- One way to do this is to relax the "desirable outcome". If we aim to construct an object that satisfies some property, maybe we can prove one can construct an object that nearly satisfies that property, and tweak the object a little in our favour afterwards. This idea is called an **alteration**.

3.2. Classical Example

Example 3.1 (Weak Turán). We call an **independent set** of a graph $G=(V,E)$ a subset of vertices $U\subset V$ such that for all $x,y\in U$, the edge does not exist, namely $(x,y)\notin E$.

Prove that if G has n vertices and an average degree of d, there exists an independent set of size at least $\frac{n}{2d}$.

Proof. Construct some set W by selecting each vertex in V with probability p (to be determined). Note that this set W will not be our independent set, but merely a step towards our independent set. It is easy to see that $\mathbb{E}[|W|]=np$ and $\mathbb{E}[\text{Edges in } W]=|E|\cdot p^2=\frac{ndp^2}{2}$. Given W, we construct an independent set by removing exactly one vertex for each edge in W. Since the newly constructed set, call it U, does not have any edges, it is an independent set. Then we have

$$\mathbb{E}[|U|]=np-\frac{ndp^2}{2}=np\cdot\left(1-\frac{dp}{2}\right),$$

and selecting $p=1/d$ yields the wanted result. Note that we cannot get a better result through this approach, as $p=1/d$ is the minimizer of the right-hand side. □

3.3. Problems

Problem 3.1 (Zarankiewicz problem). Given an integer $r>0$ and an integer $n>r$, we want to determine the largest number of edges an $n\times n$ bipartite graph can have without containing $Z_{r,r}$ (complete $r\times r$ bipartite graph) as a subgraph. We call this number $Z(n,r)$.

Prove that $Z(n,r)\geq n^{2-2/(r+1)}-\frac{n^{2r/(r+1)}}{r^2}$.

Problem 3.2. Let $G=(V,E)$ be an undirected graph on n vertices. We say $U\subset V$ is a *dominating set* if every vertex in $V\backslash U$ has at least one neighbour in U. That is, U is connected to the entire graph. Note that V is a dominating set, and that $\emptyset$ is never dominating. We are usually interested in finding small dominating sets.

Suppose the minimal degree of G is $\delta>1$. Prove that G has a dominating set of size at most $n\frac{1+\ln(\delta+1)}{\delta+1}$.

Problem 3.3 (Korea 2016/6). Let U be a set of m triangles. Prove that there exists a subset $W\subset U$ with at least $0.45\cdot m^{0.8}$ triangles, with the following property: there are no points A,B,C,D,E,F for which ABC,BCD,CDE,DEF,EFA,FAB are all in W.

Problem 3.4 (IMO 2014/6, Simplified). A set of lines in the plane is in general position if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its finite regions. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to colour at least $c\sqrt{n}$ lines blue in such a way that none of its finite regions has a completely blue boundary.

Prove the result for any $c<2/3$.

4. Inequalities

4.1. Some Theory

We recall the **Union Bound**, already introduced in Section 2.1.

Theorem 4.1 (Union Bound)**.** For any events $A_1,\ldots$ (possibly infinitely many), if

$$\mathbb{P}(A_1)+\cdots<1,$$

then none of the events occur with positive probabilty.

A different type of inequality, also very useful, is the **Markov Inequality**, which allows us to bound probabilities based on expectations.

Theorem 4.2 (Markov Inequality)**.** Let X be any random variable that takes nonnegative values only. Then it holds that

$$\mathbb{P}(X\geq\varepsilon)\leq\frac{\mathbb{E}[X]}{\varepsilon}.$$

Finally, we introduce a more complicated theorem, namely the **Local Lovasz Lemma**. The result of the Lemma is similar to the one of Union Bound, but its statement is somewhat more complicated.

Theorem 4.3 (Local Lovasz Lemma)**.** Consider events $A_1,\ldots$ (possibly infinitely many), such that $\mathbb{P}(A_i)\leq p$ for all i, and such that A_i is independent from all but at most d other events. Then, if $edp\leq 1$, it holds that none of the events occur with positive probability.

4.2. Classical Example

Example 4.4. Suppose $11n$ points are placed around a circle and colored with n different colors such that each color is applied to exactly 11 points. Show there must be a set of n points containing one point of each color but not containing any pairs of adjacent points.

Proof. Construct a set of points where we select one of each colour at random, with probability $1/11$ for each point of a given colour. Consider the events A_i that we selected the points i and $i+1$ (modulo $11n$), where $i=1,\ldots,11n$. We know that $\mathbb{P}(A_i)\leq 1/121$ (if they are of different colour it is $1/121$, else 0). Moreover, the event A_i is dependent with at most 42 other events (the points i and $i+1$ could be two different colours, in which case there are at most 21 other pairs that contain each of the colours).

In the notation of the LLL, we thus have $p=\frac{1}{121}$ and $d=42$. Since $epd=\frac{42e}{121}=0.9435\ldots<1$, it holds that none of the events occur with positive probability, which is exactly what we aimed for. □

4.3. Problems

Problem 4.1. Let $H=(V,E)$ be a hypergraph in which every edge has at least $k\geq 3$ vertices, and suppose every edge intersects at most d other edges. If $d\leq 2^{k-3}$, then H is 2-colourable, meaning it can be coloured in 2 colours without any monochromatic edges.

Problem 4.2 (USAMO 2012/6, Simplified)**.** Let $n\geq 2$ be an integer, and let $x_1,\ldots,x_n$ be real numbers such that $x_1+\cdots+x_n=0$ and $x_1^2+\cdots+x_n^2=1$. For each subset $A\subset\{1,\ldots,n\}$, we define

$$S_A=\sum_{i\in A}x_i,$$

with the convention that $S_\emptyset=0$. Prove that, for any positive number $\lambda>0$, the number of sets satisfying $S_A\geq\lambda$ is at most $2^{n-3}/\lambda^2$.

Problem 4.3 (Russia 2006, Modified)**.** At an MBL camp with possibly infinitely many participants, each participant has at least 50 and at most 100 friends among the other persons at the camp. Due to budget reasons, the camp can only afford to order C differently coloured T-Shirts. However, they have an infinite number of T-Shirts in each colour. Show that for $C=1331$, one can hand out a T-shirt to every participant such that any person has 20 friends whose T-shirts all have pairwise different colors. Is it possible to improve the value of C, to help MBL reduce its budget?

Problem 4.4 (IMO 2014/6, Simplified)**.** A set of lines in the plane is in general position if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its finite regions. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to colour at least $c\sqrt{n}$ lines blue in such a way that none of its finite regions has a completely blue boundary.

Prove the result for $c=(6e)^{-1/2}$. Note that the result in Problem 3.4 of course implies this one, but try to come up with a simpler proof for this subcase, using the inequalities approach.

5. Hints for Problems

2.1: Set $\varepsilon_k=1$ with probability 0.5, and $\varepsilon_k=-1$ with probability 0.5.

2.2: Select a tournament at random, and, for a subset A of vertices, consider the event E_A that no vertex beats all of A.

2.3: Define E_i the indicator event that number i appears (at least once) in that row or column.

2.4: Take a random permutation π of $\{1,\ldots,n\}$, and define b_i^π to be the smallest number such that $a_i\leq b_i^\pi$ satisfying $b_i^\pi\equiv\pi(i)\pmod{n}$.

2.5: Define an infinite word W, where each character is assigned randomly with probability $1/r$. Note that if two words are prefixes of W, then one must be a prefix of the other.

2.6: Select a random pairing and associate to it the number of edges that exist in the pairing. We want this number to be n for a perfect matching.

2.7: Note the sum is a polynomial in x, and hence it is sufficient to prove the equality for all $x\in[0,1]$. We can thus interpret x as a probability. Now colour the points randomly in red or blue with probability x and $1-x$. Try to interpret the polynomial as an expected value.

2.8: Define $Y = \cup_{i=1}^{h}(A_i \cup B_i)$, and consider a random order π on Y, that is a bijection $Y \to \{1,\dots,|Y|\}$, which allows us to compare any two elements $y_1, y_2 \in Y$ by the rule $y_1 \leq_\pi y_2$ if, and only if, $\pi(y_1) \leq \pi(y_2)$, where the last $\leq$ is the conventional comparison in the reals. Consider the events E_i that all elements in A_i precede all elements in B_i, according to the ordering π.

2.9: If a is irrational, then $\{\{an\} : n \in \mathbb{N}\}$ is uniformly distributed over $[0,1]$. As such, if n is chosen uniformly at random in $[1,N]$, it holds that $\mathbb{E}[\{xn\}] \to 1/2$ as $N \to \infty$. However, when a is rational, this does not hold true.

2.10: Consider a large prime $p \equiv 2 \pmod 3$, and write $p = 3k+2$. Define $C = \{k+1,\dots,2k+1\}$, which is a sum-free set modulo p. Introduce a random variable $X \in [1, p-1]$, and prove consider the random variables $X_i = Xb_i \pmod p$.

3.1: Randomly select an edge from the $n \times n$ bipartite graph, and remove an edge from edge each of $K_{r,r}$ created.

3.2: Select a set at random, where each vertex is selected with probability p. Take Y to be the set of vertices of $V \backslash X$ that have no neighbours in X. Then $U = X \cup Y$ is dominating.

3.3: Take each triangle with probability p, and for each 6-tuple inducing a set of 6 triangles, remove one triangle. Note that the property is cyclic, and preserved under symmetry.

3.4: Colour each line with probability p, and uncolour one line per completely blue polygon. You might want to distinguish between triangles and non-triangles.

4.1: Apply LLL.

4.2: Select A randomly, and compute $\mathbb{P}(S_A \geq \lambda)$ in two different ways, one being by Markov's Inequality. It might be helpful to compute $\mathbb{P}(S_A^2 \geq \lambda^2)$ instead.

4.3: Give each participant a random T-Shirt with probability $1/C$ and, for each participant P, define the event E_P to be 1 if P's friends wear at most $C-1$ different colours.

4.4: Create $c\sqrt{n}$ groups of lines, and colour one line per group.

6. Solutions

2.1 (Exercise): Let E_i be the event that baby i did not get poked. Then we are interested $E = E_1 + \cdots + E_{2006}$. We know that $\mathbb{E}[E_i] = \mathbb{P}(E_i = 1) = 0.25$, so $\mathbb{E}[E] = 501.5$.

2.2 (Exercise): Let E_i be the event that the pair standing in positions i and $i+1$ are exactly one boy and one girl. We are interested in $E = E_1 + \cdots E_{19}$. We know that $\mathbb{E}[E_i] = \mathbb{P}(E_i = 1) = \frac{7}{20} \cdot \frac{13}{19} + \frac{13}{20} \cdot \frac{7}{19} = \frac{91}{190}$. As such, $\mathbb{E}[E] = 9.1$.

2.3 (Exercise): The expected number of valid squares of the knight are $8 \cdot (1-p)$, where as the expected number of valid squares of the bishop are $4 \cdot \sum_{i=1}^{\infty}(1-p)^i = \frac{4}{p}$. It is thus sufficient to solve $2p(1-p) = 1$.

2.4 (Exercise): Take π a random permutation, such that $\mathbb{E}[\pi_i] = \frac{n+1}{2}$ for all $i = 1,\dots,n$. Denote by Π the set of all permutations, which has cardinality $n!$. By linearity of expectation, $\mathbb{E}[S(\pi)] = \frac{n+1}{2} \cdot \sum_{i=1}^{n} c_i$, and hence $\sum_{\pi \in \Pi} S(\pi) = |\Pi| \cdot \mathbb{E}[S(\pi)] = \frac{(n+1)!}{2}$ which is a multiple of $n!$ as n is odd. On the other hand, if no $S(\pi)$ are congruent to each other modulo $n!$, it must hold that

$$\sum_{\pi \in \Pi} S(\pi) \equiv \sum_{i=1}^{n!} i = \frac{(n!+1)n!}{2} \pmod{n!},$$

which is not 0 as n is odd. The contradiction shows there must exist $\pi,\mu\in\Pi$ such that $S(\pi)\equiv S(\mu) \pmod{n!}$.

2.1: Select ε_k with a coin flip. Square the wanted inequality, and use $\mathbb{E}[\varepsilon_i\varepsilon_j]=0$ for $i\neq j$, and $\mathbb{E}[\varepsilon_i^2]=1$.

2.2: Consider a random tournament on $[n]$. For every subset A of size k, let E_A be the event that no vertex beats all vertices in A. Then $\mathbb{P}(E_A=1)=(1-2^{-k})^{n-k}$, since there are $n-k$ vertices in $[n]\backslash A$, and a vertex $v\in[n]\backslash A$ does not beat all vertices in A with probability $1-2^{-k}$. Then it holds that

$$\mathbb{P}\left(\bigcup_{A\subset[n],|A|=k} E_A\right)\leq \sum_{A\subset[n],|A|=k}\mathbb{P}(E_A)=\binom{n}{k}(1-2^{-k})^{n-k}<1.$$

As such, the probability that all events E_A occur is <1, so there must exist a tournament where no E_A occur, which is equivalent to property S_k.

2.3: Select a row or a column uniformly at random (total of $2n^2$ scenarios). Define E_i the event that number i appears (at least once) in that row or column. Observe that $\mathbb{E}[E_i]=\mathbb{P}(E_i=1)\geq\frac{1}{n}$ (worst case scenario is when all occurrences are clustered in a $n\times n$ submatrix), such that $\mathbb{E}[E_1+\cdots+E_{n^2}]\geq n$.

2.4: Firstly, note that if $a_i>n$, we can replace both a_i and b_i by a_i-n and b_i-n. As such, we may suppose without loss of generality that $a_i\in[1,n]$.

Pick a random permutation π of $\{1,\dots,n\}$ and define $b_i^\pi=n+\pi(i)$ if $a_i>\pi(i)$, and $b_i^\pi=\pi(i)$ else. By construction, properties 1 and 2 are satisfied for all permutations π.

Define A_π to be the number of indices such that $a_i>\pi(i)$. Then it holds that $\sum_{i=1}^n b_i=\sum_{i=1}^n\pi(i)+A_\pi n=\frac{n(n+1)}{2}+A_\pi n$. By linearity of the expectation,

$$\mathbb{E}[A_\pi]=\sum_{i=1}^n\mathbb{P}(a_i>\pi(i))=\sum_{i=1}^n\frac{a_i-1}{n}=\sum_{i=1}^n\frac{a_i}{n}-1,$$

such that there exists a π such that $A_\pi\leq\left\lfloor\sum_{i=1}^n\frac{a_i}{n}-1\right\rfloor$. For that choice of π,

$$\sum_{i=1}^n b_i^\pi\leq\frac{n(n+1)}{2}+n\left\lfloor\sum_{i=1}^n\frac{a_i}{n}\right\rfloor-n,$$

as wanted.

2.5: Define an infinite word W over the alphabet, where each character is assigned randomly with probability $1/r$. For a given word w, define the indicator event A_w to be 1 is w is a prefix of W, and 0 elsewise. Then $\mathbb{E}[A_w]=\mathbb{P}(A_w=1)=r^{-|w|}$. As such, by assumption, $\mathbb{E}[\sum_{w\in\mathcal{F}}A_w]>1$, and since $\sum_{w\in\mathcal{F}}A_w\in\mathbb{Z}$, it there must exist a word W such that $\sum_{w\in\mathcal{F}}A_w\geq 2$. As such there exist two distinct words $w,v\in\mathcal{F}$ both prefixes of W, meaning one must be a prefix of the other.

2.6: Select a random permutation $\pi\colon[n]\to[n]$ of $[n]$, and create the pairing where we associate vertex $a_i\in V_1$ to vertex $b_{\pi(i)}\in V_2$. We associate to such a pairing a score, which is the number of edges that do exist. We want to show there exists a pairing with score n. We define E_i the event that the edge between a_i and $b_{\pi(i)}$ exists. Then $\mathbb{E}[E_i]=\mathbb{P}(E_i=1)=\deg(a_i)/n$. Moreover, we are interested in $E=E_1+\cdots+E_n$, where

$$\mathbb{E}[E]=\mathbb{E}[E_1]+\cdots+\mathbb{E}[E_n]=\frac{\deg(a_1)+\cdots+\deg(a_n)}{n}\geq\frac{n^2-n+1}{n}=n-\frac{n-1}{n}>n-1.$$

As E counts the number of good pairs, it must be an integer, and hence there must exist a permutation (giving rise to a pairing) with score n, as wanted.

2.7: Colour each point red or blue with probability x and $1-x$, and define E_P the indicator event that all vertices of P are red while all of its exterior points are blue. Then $\mathbb{E}[E_P]=\mathbb{P}(E_P=1)=x^{a(P)}(1-x)^{b(P)}$. The polynomial is exactly the expected number of polygons P such that E_P occurs. To conclude, note exactly one of the events must occur.

2.8: Let $Y=\cup_{i=1}^{h}(A_i\cup B_i)$ be all the elements appearing in any of the subsets. Consider a random order π on Y, that is a bijection $Y\to\{1,\dots,|Y|\}$, which allows us to compare any two elements $y_1,y_2\in Y$ by the rule $y_1\leq_\pi y_2$ if, and only if, $\pi(y_1)\leq\pi(y_2)$, where the last $\leq$ is the conventional comparison in the reals.

Now define E_i the event that all elements in A_i precede all elements in B_i according to the ordering π. Note that E_i are all pairwise disjoint. Indeed, suppose E_i and E_j occur with $i\neq j$. By symmetry, we can suppose the last element (according to the ordering π) of A_j is not after the last element (still according to π) of A_i. However, since all elements of B_i come after all elements of A_i, and hence also after all elements of A_j, it holds that $A_j\cap B_i=\emptyset$. This is a contradiction, so E_i and E_j cannot occur at the same time, as wanted.

Moreover, we note that $\mathbb{P}(E_i)=1/\binom{k+l}{k}$, which may be seen easily by restricting the ordering π to the set $A_i\cup B_i$. As such, by the equality case of Boole's Inequality, we obtain

$$1\geq\mathbb{P}(\cup_{i=1}^{h}E_i)=\sum_{i=1}^{h}\mathbb{P}(E_i)=h/\binom{k+l}{k},$$

as wanted.

2.9: Suppose none is integer. Since $\lim_{n\to\infty}\lfloor an\rfloor/n=a$, we deduce $a+b=c$, which shows that $\{an\}+\{bn\}=\{cn\}$. The hint tells us that $\mathbb{E}[\{xn\}]\to 1/2$ as $N\to\infty$ if x is irrational and n is chosen uniformly at random in $[1,N]$. If however $x=p/q$ with $\gcd(p,q)>1$ and $q>1$, one can compute $\mathbb{E}[\{xn\}]\to\frac{q-1}{2q}=\frac{1}{2}-\frac{1}{2q}$ if n is uniformly chosen at random in $[1,N]$ and $N\to\infty$. In specific, it always holds that $\mathbb{E}[\{xn\}]\to t\in[1/4,1/2]$ as $N\to\infty$, where $t=1/2$ if, and only if, x is irrational. Now take expectations on the equality $\{an\}+\{bn\}=\{cn\}$ and limits as $N\to\infty$ to deduce that $t_a+t_b=t_c$, where $t_a,t_b,t_c\in[1/4,1/2]$. The only possibility is $t_a=1/4, t_b=1/4$ and $t_c=1/2$, which implies a,b are rational and c is irrational, which contradicts $a+b=c$.

2.10: Call the set $B=\{b_1,\dots,b_n\}$.

Let p be a large prime number (say larger than $10\cdot\max_i|b_i|$) satisfying $p\equiv 2 \pmod 3$, and write $p=3k+2$. Define $C=\{k+1,\dots,2k+1\}$ a set of $k+1$ elements, which we note is sum-free modulo p.

Choose $X\in[1,p-1]$ a random integer, uniformly at random (so $X=x$ for $x\in[1,p-1]$ with probability $1/(p-1)$). Define the random variables $X_i\in[1,p-1]$ such that $X_i\equiv Xb_i \pmod p$. Since all possible values of X are invertible modulo p, as X varies over $[1,p-1]$, X_i varies over $[1,p-1]$ as well. As such, for each i,

$$\mathbb{P}(X_i\in C)=\frac{|C|}{p-1}=\frac{k+1}{3k+1}>\frac{1}{3}.$$

As such, the expected number of elements X_i in C is strictly larger than $\frac{n}{3}$. Note that such a subset $A'\subset C$ naturally induces a sum-free subset of B, as all elements in A' will be of the form

xb_i for some x, and that x is invertible modulo p, and that all b_i are much smaller than p. As such, $A=\{x^{-1}a_i\colon a_i\in A'\}$ will be a sum-free subset of B.

3.1: Pick each edge in the $n\times n$ bipartite graph with probability p, to be determined. The expected number of edges is then n^2p. Moreover, the expected number of $K_{r,r}$ contained in it is $\binom{n}{r}^2 p^{r^2}$. Perform an alteration removing one edge from each $K_{r,r}$, such that, after the alteration, we have

$$\mathbb{E}[\text{Edges}]=n^2p-\binom{n}{r}^2 p^{r^2}\geq n^2p-\frac{n^{2r}}{(r!)^2}p^{r^2}\geq n^2p-\frac{n^{2r}}{r^2}p^{r^2}.$$

The value of p that minimizes the right-hand side is $p=n^{-2/(r+1)}$, which yields

$$\mathbb{E}[\text{Edges}]\geq n^{2-2/(r+1)}-\frac{n^{2r-2r^2/(r+1)}}{r^2}=n^{2-2/(r+1)}-\frac{n^{2r/(r+1)}}{r^2},$$

as wanted.

3.2: We select a set X randomly, by including every vertex $v\in V$ into X with probability p, where $p\in[0,1]$ will be determined later. Let Y be the set of vertices in $V\backslash X$ that do not have a neighbour in X. We want to show that Y can be empty for a small enough set X. Firstly, note that $\mathbb{E}[|X|]=np$ and that $\mathbb{P}(v\in Y)\leq(1-p)^{\delta+1}$ (the probability that $v\in Y$ is the probability that v and all its neighbours (at least δ of them) are not in U). Then it holds that

$$\mathbb{E}[|X|+|Y|]=\sum_{v\in V}\mathbb{E}[\mathbf{1}(v\in X)]+\sum_{v\in V}\mathbb{E}[\mathbf{1}(v\in Y)]=\sum_{v\in V}\mathbb{P}(v\in X)+\sum_{v\in V}\mathbb{P}(v\in Y)\leq np+n(1-p)^{\delta+1}.$$

Now note that $1-p\leq e^{-p}$, and hence

$$\mathbb{E}[|X\cup Y|]=\mathbb{E}[|X|+|Y|]\leq np+ne^{-p(\delta+1)}.$$

Clearly $U=X\cup Y$ is a dominating set, and by selecting $p=\frac{\ln(\delta+1)}{\delta+1}$ we obtain the wanted bound.

3.3: We say a tuple of 6 points (A,B,C,D,E,F) is "bad" it satisfies the property. Note that there are at most $m\cdot(m-1)\cdot(3!)^2=36m(m-1)$ bad tuples. We shall identify tuples that induce the same triangles. Specifically, circular permutations (total of 6) and symmetric permutations (multiples by 2) induce the same tuples. As such, the number of "bad" tuples inducing unique triangles is at most $3m(m-1)$.

Choose a subset $V\subset U$ randomly, each triangle with a probability p. Then $\mathbb{E}[|V|]=p\cdot|U|=mp$. The expected number of "bad" sets is

$$\mathbb{E}[\text{“Bad” sets in }V]\leq 3m(m-1)\cdot p^6.$$

We now fix V by removing a triangle from each "bad" tuple, such that no "bad" tuples remain, and call this new set W. Then

$$\mathbb{E}[|W|]=mp-3m(m-1)\cdot p^6\geq mp-3m^2p^6.$$

We note that the right-hand side is minimized for $p=(18m)^{-1/5}$, yielding

$$\mathbb{E}[|W|]\geq\frac{m^{4/5}}{18^{1/5}}-\frac{m^{4/5}}{6\cdot 18^{1/5}}=m^{0.8}\cdot\frac{5}{6\cdot 18^{1/5}}\geq 0.45\cdot m^{0.8},$$

as wanted.

3.4: Firstly, we observe there are at most $\frac{1}{3}n^2$ triangles (any vertex can be part of at most two triangles since the lines are in general position, and a triangle has three vertices, so there are at most $\frac{2}{3}\binom{n}{2} < \frac{1}{3}n^2$ triangles). Using Euler's formula $V - E + F = 2$ with $V = \binom{n}{2}$ and $E = n(n-2)$, we get that $F < \frac{1}{2}n^2$ (for $n \geq 2$).

Now colour each line in blue with probability p (to be determined). Clearly $\mathbb{E}[\text{Lines}] = np$, and $\mathbb{E}[\text{Blue Triangles}] \leq p^3 \cdot \frac{1}{3}n^2$. Moreover, there are at most (large bound) $\frac{1}{2}n^2$ polygons with more than 4 sides, so

$$\mathbb{E}[\text{Blue Non-Triangle Polygons}] < p^4 \cdot \frac{1}{2}n^2.$$

We now remove some blue lines (we perform an alteration), namely in each region that is completely blue, we simply uncolour one line. After this alternation, we have

$$\mathbb{E}[\text{Blue Lines}] = np - p^3 \cdot \frac{1}{3}n^2 - p^4 \cdot \frac{1}{2}n^2.$$

Now select $p = 1/\sqrt{n}$, such that we have

$$\mathbb{E}[\text{Blue Lines}] = \sqrt{n} - \frac{1}{3}\sqrt{n} - \frac{1}{2} = \frac{2}{3}\sqrt{n} - \frac{1}{2}.$$

Since $c < 2/3$, if n is very large, we have $\mathbb{E}[\text{Blue Lines}] \geq c\sqrt{n}$, as desired.

4.1: Colour each vertex randomly and independently in blue or red, with probability 1/2. For each edge $e \in E$, define A_e the event that e is monochromatic. Then $\mathbb{P}(A_e) = 2/2^{|e|} \leq 2^{1-k} = p$. Moreover, A_e depends on at most d other events, namely the events induced by the edges that e intersects. Since $edp \leq 4dp \leq 1$ by assumption, we conclude by LLL.

4.2: Select A randomly, by including $i \in \{1, \ldots, n\}$ into A with probability 1/2. Denote A_i a random variable to be 1 if $i \in A$, and 0 otherwise. Then it holds that $S_A = \sum_{i=1}^{n} A_i x_i$. As such,

$$\mathbb{E}[S_A^2] = \mathbb{E}\left[\left(\sum_{i=1}^{n} A_i x_i\right)^2\right] = \sum_{i=1}^{n} \mathbb{E}[A_i^2] x_i^2 + 2 \sum_{i,j=1, i<j}^{n} \mathbb{E}[A_i A_j] x_i x_j.$$

One easily computes $\mathbb{E}[A_i^2] = \frac{1}{2}$ and $\mathbb{E}[A_i A_j] = \frac{1}{4}$, such that

$$\mathbb{E}[S_A^2] = \frac{1}{2}\sum_{i=1}^{n} x_i^2 + \frac{1}{2}\sum_{i,j=1, i<j}^{n} x_i x_j = \frac{1}{2} + \frac{1}{4}\left(\left(\sum_{i=1}^{n} x_i\right)^2 - \sum_{i=1}^{n} x_i^2\right) = \frac{1}{4}.$$

As such, by Markov's Inequality,

$$\frac{|\{A \colon S_A \geq \lambda\}|}{2^n} = \mathbb{P}(S_A \geq \lambda) = \frac{1}{2}\mathbb{P}(S_A^2 \geq \lambda^2) \leq \frac{\mathbb{E}[S_A^2]}{2\lambda^2} = \frac{1}{8\lambda^2},$$

as wanted.

4.3: Give each participant a random T-Shirt with probability $1/C$. For each participant P, define the event E_P to be 1 if P's friends wear at most $C-1$ different colours. We wish to prove that with positive probability none of these events happen.

If two participants A and B are not friends and don't have a mutual friend, the events E_A and E_B are independent. As such, E_P is dependent on at most $d = 100 + 99 \cdot 100 = 10000$ other events.

Moreover, it holds that, for any participant P with $k \in [50, 100]$ friends,

$$\mathbb{P}(E_P) = \binom{C}{19} \cdot \left(\frac{19}{C}\right)^k \leq \frac{C^{19}}{19!} \cdot \frac{19^{50}}{C^{50}} \leq \frac{19^{50}}{19! \cdot C^{31}} = p.$$

So we can conclude by LLL if $edp \leq 1$, which is clearly the case for $C = 1331$, and still holds for $C = 46$ (although this is hard to see without a calculator).

4.4: Split the n lines into $c\sqrt{n}$ groups of size $\sqrt{n}/c$ each, randomly. For each group, we select one to be coloured blue, at random.

Denote all the regions by indices $k = 1, \ldots, K$, and define the events E_k to be 1 if and only if three of the bounding lines of region k are coloured in blue. We want to show there is a positive probability that none of the events E_k occur. Note that $\mathbb{P}(E_k) \leq (c\sqrt{n})^3$ (All are blue with probability $(c\sqrt{n})^3$ if they are all in separate groups of lines, else the probability is 0). So denote $p = (c\sqrt{n})^3$.

Moreover, a line is part of $2(n-1)$ regions (one on each side of the line, and other lines intersect it $n-1$ times). 4 of these regions will be infinite (the ones at each end), hence a line is part of $2n-6$ finite regions. As such, since each group of lines has $\sqrt{n}/c$ lines, each event depends on at most $d = 3 \cdot (2n-7) \cdot \sqrt{n}/c$ other events.

Now notice that for $c = (6e)^{-1/2}$, it holds that

$$edp = e \cdot 3 \cdot (2n-7) \cdot \sqrt{n}/c \cdot (c\sqrt{n})^3 = \frac{3e \cdot (2n-7) \cdot c^2}{n} \leq 6e \cdot c^2 \leq 1,$$

and hence LLL guarantees there is a positive probability none of the events hold, as wanted.

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